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Piotr T. Chrusciel, Malcolm A. H. MacCallum and David B. Singleton

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Gravitational waves in general relativity XIV. Bondi expansions and the 'polyhomogeneity' of $\mathscr{I}$

By Piotr T. Chruściel[1], Malcolm A. H. MacCallum[2] and David B. Singleton[3]

[1] *Max Planck Institut für Astrophysik, Karl Schwarzschild Strasse 1, D–85740 Garching bei München, F.R.G.*
[2] *School of Mathematical Sciences, Queen Mary and Westfield College, University of London, Mile End Road, London E1 4NS, U.K.*
[3] *Supercomputer Facility, Australian National University, Canberra A.C.T. 2601, Australia*

The structure of *polyhomogeneous* space-times (i.e. space-times with metrics which admit an expansion in terms of $r^{-j}\log^i r$) constructed by a Bondi–Sachs type method is analysed. The occurrence of some log terms in an asymptotic expansion of the metric is related to the non-vanishing of the Weyl tensor at $\mathscr{I}$. The validity in this more general context of various results from the standard treatment of $\mathscr{I}$, including the Bondi mass loss formula, the peeling-off of the Riemann tensor and the Newman–Penrose constants of motion, is considered.

1. Introduction

In general relativity an important question is: what does the gravitational field of a radiating asymptotically Minkowskian system look like? The answer to that question proposed by Bondi *et al.* (1962), Sachs (1962) and Penrose (1965) seems to have been adopted by researchers (cf. Wald 1984; Newman & Tod 1980), in spite of the wide evidence against this proposal: indeed it has been suggested both by the analysis of Christodoulou & Klainerman (1993) and by various approximate calculations (cf. Damour (1986) and references therein) that such systems generically do not satisfy the Bondi–Penrose–Sachs asymptotic conditions. In a recent study (Andersson & Chruściel 1994*a*; cf. also Andersson & Chruściel 1993; Andersson *et al.* 1992) of the asymptotic properties of solutions of constraint equations on spacelike hypersurfaces intersecting '$\mathscr{I}$' transversally it has similarly been observed that generic Cauchy data constructed in such a setting by the 'conformal method' failed to be smoothly extendable, after appropriate rescalings, to the conformal boundary. More precisely, it has been shown (cf. Andersson & Chruściel 1994*a*, *b* for more details) that, when considering Cauchy data constructed by the conformal method with smooth up to boundary 'seed' fields and with the condition $\operatorname{tr} K = \text{const.} \neq 0$, one has:

(i) generically, for such data no conformal factor Ω exists for which the shear of $\mathscr{I}$ in the metric $\Omega^2\gamma$ vanishes at $\partial\Sigma$; by the vanishing of the shear of $\mathscr{I}$ we

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mean the somewhat stronger statement that

$$\nabla_\mu \nabla_\nu \Omega \Big|_{\mathscr{I}} = 0 \tag{1.1}$$

(recall that in the case of $C^2(\bar{\mathcal{M}})$ metrics the existence of an Ω such that (1.1) holds follows from the vacuum field equations (cf. Wald 1984));

(ii) consider those data for which the shear of $\mathscr{I}$ vanishes for appropriately chosen Ω. Generically, for such data the Weyl tensor of $\Omega^2\gamma$ *does not* vanish at $\mathscr{I}$. (Recall that for vacuum metrics γ such that $\Omega^2\gamma$ is C^3 up to boundary on $\tilde{\mathcal{M}}$ the vanishing of the Weyl tensor of $\Omega^2\gamma$ at $\mathscr{I}$ follows by a theorem of Penrose (Penrose 1965; Penrose & Rindler 1986; Geroch 1977).)

The results obtained by Andersson & Chruściel (1994*a*) seem to indicate very strongly that a consistent set-up in which the gravitational radiation field can be described in *generic* situations is that of manifolds $(\tilde{\mathcal{M}}, \tilde{\gamma})$, $\tilde{\gamma} = \Omega^2\gamma$ with metrics $\tilde{\gamma}$ which are not smooth but *polyhomogeneous*† near $\mathscr{I}$. (A function f is called *polyhomogeneous* if it admits an expansion in terms of $r^{-j}\log^i r$ rather than r^{-j}, cf. Appendix A for a more precise definition.) The object of this paper is to show that at least part of the results described above can be obtained in a rather simpler way in a Bondi–Sachs type setting, as set out in earlier papers in this series (Bondi *et al.* 1962; Sachs 1962; van der Burg 1966; these are referred to here as Papers VII, VIII and IX, respectively).

In §2, we show that the hypothesis of polyhomogeneity of $\mathscr{I}$ is *formally consistent* with the Einstein equations. (Note, however, that thanks to the important theorems of Friedrich (1986, 1988, 1991), together with the results of Andersson & Chruściel (1994*a*) and Andersson *et al.* (1992), a large class of space-times satisfying the Bondi–Penrose–Sachs conditions is now known to exist. On the other hand no proof that the Cauchy problem is well posed for polyhomogeneous but not smooth initial data of 'hyperboloidal' type is available yet.) We show that the characteristic initial value problem of Bondi–Sachs type is formally well posed in the space of polyhomogeneous metrics, in the sense that the (retarded) time derivatives of the fields on the initial data hypersurface are polyhomogeneous if the free initial data are (with the same 'degrees of polyhomogeneity' when these degrees are chosen appropriately; cf. §2 for details), and, in a manner completely analogous to that of the original Bondi–Sachs analysis, that one can write down a hierarchy of evolution equations for the coefficients of the polyhomogeneous expansion of the free data.

In §3 we show that in the class of space-times considered in this paper the conformal factor Ω can always be chosen so that (1.1) holds. Thus, Cauchy data incompatible with (1.1) cannot lead to a space-time of the type considered here (cf. Andersson & Chruściel (1994*b*) for a similar result in a somewhat different setting). We prove that initial data, constructed by a Bondi–Sachs procedure starting from free data smooth at $\mathscr{I}$, will be smooth at $\mathscr{I}$ if and only if the free initial data are such that the Weyl tensor of $\Omega^2\gamma$ vanishes at $\mathscr{I}$. We find that the Trautman–Bondi mass loss formula (Trautman 1958; Bondi *et al.* 1962;

† The term *polyhomogeneous* seems to have been adopted in the mathematical literature for the kind of expansion considered here (cf. Mazzeo 1991). Gel'fand & Shilov (1964) use the term *associated homogeneous* for a similar notion. The members of the Garching relativity seminar have suggested use of the term *polylogarithmic* for this kind of expansion. Winicour (1985) uses the term *logarithmic asymptotic flatness* in a somewhat similar setting.

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Sachs 1962) remains unchanged in the polyhomogeneous case; thus the Bondi mass is still well defined, and is a monotonically decreasing function of retarded time‡. We also note that for a class of polyhomogeneous metrics the peeling-off property of the Riemann tensor is the same as the one for smooth metrics up to $O(r^{-2-\epsilon})$, $0 \leqslant \epsilon < 1$, terms. We show that some quantities built out of the restriction of the Weyl tensor to $\mathscr{I}$ are (in general nontrivial) constants of motion, as already noted by Winicour (1985) and by Christodoulou & Klainerman (1993). More generally, we find (in § 2) that the 'leading log coefficients' of the polyhomogeneous expansion are constants of motion. We argue, from an explicit calculation in the axisymmetric case, that the Newman–Penrose constants of motion (Newman & Penrose 1965; Penrose & Rindler 1986) cease to be constants of motion in generic polyhomogeneous situations, although our example does give a new constant of the motion. (It could, therefore, be that some new functionals of the field, which reduce to the Newman–Penrose constants of motion when $\mathscr{I}$ is smooth, are constants of motion in the polyhomogeneous situation. We do not have an answer to that question.) Section 4 considers the construction of Bondi coordinates in our more general setting.

The results of our analysis show that the presence of some $\log r$ terms in an asymptotic expansion of the metric is quite natural, and does not lead to any serious extra difficulties in the analysis of the geometry. Recall that the imposition of the conditions which lead to the vanishing of the log terms was interpreted in some earlier papers of this series as an *outgoing radiation condition* (Bondi *et al.* 1962; Sachs 1962; van der Burg 1966). Two concerns had to be addressed: the possibility of advanced rather than retarded solutions, and the possibility of retarded waves travelling in the inward radial direction but at indefinitely large distances. With the help of our present understanding of $\mathscr{I}$, it is clear that if $\mathscr{I}^+$ is well-defined, as it is here, there is no advanced wave involved, and that a space-time has purely outgoing radiation if and only if there is no radiation at $\mathscr{I}^-$ (cf. also Leipold & Walker (1977) for a similar point of view and for explanation of the difficulties that arise with local characterization of incoming and outgoing parts of the field even for linear theories in flat space). Since the space-times discussed here can satisfy both these requirements, we can safely abandon the 'outgoing radiation' condition of Bondi *et al.* (1962).

Moreover, we note that there exists a family of electrovacuum 'small data' space-times constructed by Cutler & Wald (1989), and also a family of 'small data' Einstein–Yang–Mills spherically symmetric space-times constructed by Bartnik (1992), which have the following properties: they possess a smooth $\mathscr{I}^+$ and a smooth $\mathscr{I}^-$, and decay to a smooth i^+ in the future and a smooth i^- in the past. Because the metric decays smoothly both in the future and the past there is both *outgoing* and *incoming* radiation in those space-times. Since both $\mathscr{I}$s are smooth, the 'outgoing radiation condition' holds at $\mathscr{I}^+$, and an analogous 'in-

‡ More precisely, for all polyhomogeneous metrics there is a quantity which we call the Bondi mass, which is a nonincreasing function of retarded time, and which reduces to the quantity defined by Bondi when Bondi's hypotheses are satisfied. We believe that the 'real mass' should not be defined *ad hoc*, but by a limiting procedure involving perhaps the Freud 'superpotential' for Einstein's energy, as done e.g. by Trautman (1958). If one does that, we expect that one will find equality of the quantity we define as the Bondi mass with the quantity obtained from the limiting procedure only when V has *no* $\log^N$ terms, i.e. when the logarithmic terms in V start at the r^{-i} level, with some $i \geqslant 1$. A precise formulation of such statements lies outside the scope of this paper.

coming radiation condition' holds at $\mathscr{I}^-$, which is clearly absurd. These examples show that not only is the vanishing of the log terms at $\mathscr{I}$ unnecessary to ensure 'outgoing radiation', it also does not prevent incoming radiation. We conclude that the association of the absence of $\log r$ terms with 'outgoing radiation' lacks justification, and that a better formulation of an 'outgoing radiation' condition could probably be given in terms of constancy of the Bondi mass at $\mathscr{I}^-$.

In the discussion above we have adopted what we consider to be now a standard notion of 'incoming' and 'outgoing' radiation. In particular, it is clear, from the hyperbolic nature of the Einstein equations, that whenever a conformal completion of the space-time exists in which the conformal boundary is an *incoming null topological surface*, then there can be *no influx* of gravitational radiation (or, for that matter, of any non-tachyonic matter fields) through the surface in question. Thus, the existence of a conformal completion of the above described nature guarantees that we have an isolated system evolving in a self-consistent way, regardless of whether or not the fields are asymptotically Minkowskian†, regardless of the decay rates of the fields towards the conformal boundary, the degrees of differentiability of some perhaps conformally rescaled fields at the conformal boundary, etc. In view of that observation it might not be so surprising that for the polyhomogeneous $\mathscr{I}$s considered here the Trautman–Bondi mass-loss law holds, regardless of the occurrence of some perhaps high powers of $\log r$ in the $1/r$ terms in the metric.

It should be pointed out that several of the results discussed here have already been observed in a similar setting by Winicour (1985). (However, we learned about this paper only after most of the work presented here was completed. Also it seems that in Winicour (1985) emphasis is put on somewhat different issues.) In this context one should also mention the results of Novak & Goldberg (1982) (cf. also Couch & Torrence 1972; Moreschi 1987), who perform a somewhat similar analysis of the Newman–Penrose equations on a null initial hypersurface. We have been informed by R. J. Torrence and W. E. Couch that the occurrence of log terms and the associated constants of motion had been observed by them a long time ago‡. Recently polyhomogeneity of null infinity has also been observed for cylindrically symmetric space-times in Ashtekar *et al.* (1994), and for infinite rotating discs in Bičák *et al.* (1993).

The approximate results reviewed by Damour (1986) referred to earlier include several papers in which logarithmic terms similar to those we consider appear. In addition there are early results of Bonnor & Fock (1957, 1959) in which logarithmic terms arose (only to be removed, in Bonnor (1959), by a coordinate transformation). To give a full analysis of those papers in the more rigorous spirit of the present work would be an interesting but lengthy task, so we note only that, for example, such terms also did not arise in the further work on the double series approximation (Bonnor & Rotenberg 1966).

We leave it an open question which precisely of the log terms which appear in our expansions can be removed by a coordinate transformation. It should,

† As pointed out below, several results proved in this paper (in particular the self-consistency of the polyhomogeneous set-up) will still be true if the 'sphere of null directions' S^2 is replaced by an arbitrary two-dimensional, perhaps but not necessarily compact, manifold M^2. Examples of vacuum space-times with such an asymptotic structure (and actually a smooth $\mathscr{I}$) are given by, for example, some Robinson–Trautman space-times.

‡ R. J. Torrence and W. E. Couch, unpublished. Personal communication from R. J. Torrence.

however, be pointed out that all the log terms have a geometric character on a *fixed* initial hypersurface, in the sense that the Bondi coordinates on a fixed hypersurface are uniquely defined. Moreover it follows by function-counting that most of the log terms cannot be removed by deforming¶ the hypersurfaces $u =$ const., as those deformations can be parametrized by the single function α at $\mathscr{I}$, cf. § 4. Moreover, for purely geometric reasons, it should also be emphasized that those log terms which we have tied to the non-vanishing of the Weyl tensor at $\mathscr{I}$ *cannot* be removed by a coordinate change.

2. The Bondi–Sachs characteristic initial value problem

In this section we shall consider the initial value problem for space-times $(\mathcal{M}, \gamma)$ with a metric of the form

$$\gamma_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -\frac{V\mathrm{e}^{2\beta}}{r}\,\mathrm{d}u^2 - 2\mathrm{e}^{2\beta}\,\mathrm{d}u\,\mathrm{d}r + r^2\,h_{ab}(\mathrm{d}x^a + U^a\mathrm{d}u)(\mathrm{d}x^b + U^b\mathrm{d}u). \quad (2.1)$$

We shall mainly be interested in the behaviour of γ on the hypersurface†

$$\mathcal{N} = \{u = 0,\ r \geqslant R,\ x^a \in S^2\},$$

where S^2 is topologically a two-dimensional sphere. (The question of the existence of coordinate systems in which an asymptotically Minkowskian metric takes the form (2.1) is considered in § 4.) As has been analysed by Bondi *et al.* (1962) in the axisymmetric case and by Sachs (1962) in general (cf. also van der Burg (1966)), to construct a vacuum metric of the form (2.1) one has to prescribe on $\mathcal{N}$ the family of metrics $h(r) \equiv h_{ab}(r, x^a)\,\mathrm{d}x^a\,\mathrm{d}x^b$ on S^2 parametrized by r, the family of vector fields $U(r) \equiv U^a(r, x^a)\partial_a$ on S^2 parametrized by r, and the scalar fields V and β. These quantities are not freely specifiable but have to satisfy constraint equations:

$$\forall \quad X \in T\mathcal{N}, \qquad R_{\mu\nu}\,k^\mu\,X^\nu = 0, \quad (2.2)$$

where k^μ is any null vector field tangent to $\mathcal{N}$ (e.g. $k^\mu\partial_\mu = \partial_r$). As has been emphasized in Bondi *et al.* (1962) and Sachs (1962), the equations (2.2) do not impose any restrictions on $h_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b$, and in fact can be viewed as equations which together with appropriate boundary conditions determine V, β and $U^a\,\partial_a$ given $h_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b$. In Bondi *et al.* (1962) and Sachs (1962) it was shown that if we assume $h_{ab} \in C^\infty(\bar{\mathcal{N}})$ and moreover

$$h_{ab}(r, x^a) = \hat{h}_{ab}(x^a) + \frac{h^1_{ab}(x^a)}{r} + \frac{a(x^a)\hat{h}_{ab}(x^a)}{r^2} + O\left(\frac{1}{r^3}\right), \quad (2.3)$$

for some functions $\hat{h}_{ab}(x^a), h^1_{ab}(x^a), a(x^a)$, then we will obtain

$$r^{-2}V, \beta, U^a, \frac{\partial h_{ab}}{\partial u} \in C^\infty(\bar{\mathcal{N}}).$$

In Bondi *et al.* (1962) and Sachs (1962) the absence of trace-free r^{-2} terms

¶ This argument is valid of course only if one does *not* make any supplementary hypotheses, such as staticity of the space-time in the far past, etc.

† Most of the analysis presented here goes through when S^2 is replaced by any two-dimensional, compact, orientable manifold ${}^2\mathcal{M}$.

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in (2.3) was termed the 'outgoing wave condition'. This condition was imposed *a priori* in Bondi *et al.* (1962) and Sachs (1962) because the occurrence of the trace-free terms led to $r^{-j}\log^i r$ terms in U^a, V and subsequently in $\partial h_{ab}/\partial u$ – this in turn led to $r^{-j}\log^i r$ terms in h_{ab} at any later moment of time. It is therefore clear that a correct set-up for analysing the characteristic initial value problem for metrics of the form (2.1) is that of metrics $h_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b$ which are *polyhomogeneous* to start with (i.e. admit an asymptotic expansion in terms of $r^{-j}\log^i r$). It is the aim of this paper to re-examine both the constraint and the evolution equations for metrics of the form (2.1) in a polyhomogeneous set-up.

Before proceeding to a detailed analysis of the Einstein equations, let us first consider the question of the boundary conditions satisfied by the fields under consideration. Let therefore a polyhomogeneous metric $h_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b$ (see Appendix A for precise definitions) be given, and suppose moreover that $h_{ab}\in C^0(\bar{\mathcal{N}})$ (if we write $h_{ab}\in\mathcal{A}^{\{N_i\}}$, then the hypothesis $h_{ab}\in C^0(\bar{\mathcal{N}})$ is equivalent to the condition $N_0=0$). It follows that the limits

$$\hat{h}_{ab}=\lim_{r\to\infty} h_{ab}$$

exist, with $\hat{h}_{ab}\in C^\infty(S^2)$. As is well known, (cf. Christodoulou & Klainerman (1993) for a simple and elegant proof) there exists a diffeomorphism $\Phi : S^2\to S^2$ such that we have $\Phi^*\hat{h}=\phi^2\breve{h}$, where $0<\phi\in C^\infty(S^2)$ and $\breve{h}$ is the standard round metric on S^2,

$$\breve{h}_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b=\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\varphi^2. \tag{2.4}$$

Replacing (r,x^a) by $(\bar{r},\bar{x}^a)=(\phi r,\Phi^a(x^b))$ one obtains a metric of the form (2.1) in which (dropping bars on $\bar{r},\bar{x}^a$)

$$\lim_{r\to\infty} h_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b=\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\varphi^2. \tag{2.5}$$

It is not too difficult to show from equations (2.2) (which are written out in detail in Appendix C) that under the condition $h_{ab}\in\mathcal{A}_{phg}$ the limits,

$$H\equiv\lim_{r\to\infty}\beta, \tag{2.6}$$

$$X^a\equiv-\lim_{r\to\infty}U^a, \tag{2.7}$$

exist, with $H, X^a\in C^\infty(S^2)$. Suppose for a moment that there actually exists a space-time with a metric of the form (2.1) on a set $U_\epsilon\equiv\{u\in(-\epsilon,\epsilon),\ r>R,\ x^a\in S^2\}$ with some $\epsilon>0$. (We should stress that in all the results obtained in this section the hypothesis of the existence of an evolution of the initial data defined on U_ϵ for some $\epsilon>0$ is *not necessary*. This is due to the fact that all our analysis involves only equations on $\mathcal{N}$). Let $\psi(u,x^a)$ be the one parameter family of diffeomorphisms of S^2 generated by the vector field $X=X^a\,\partial_a$; we thus have

$$\psi^a(u,x^b)|_{u=0}=x^a,$$

$$\partial\psi^a(u,x^b)/\partial u=X^a(u,\psi^c(u,x^b)).$$

It is easily seen that the coordinate transformation $(u,r,x^a)\to(\bar{u},\bar{r},\bar{x}^a)=(u,r,\bar{x}^a)$, with $\bar{x}^a$ implicitly defined by $x^a=\psi^a(u,\bar{x}^b)$, leads to a metric of the form (2.1) for which we have

$$\lim_{r\to\infty} U^a = 0. \tag{2.8}$$

The above argument shows that the vector field $X^a\,\partial_a$ defined by (2.7) has a gauge character, at least from a four-dimensional point of view. It should also be pointed out that the transformations leading to (2.5) and (2.8) are compatible with an initial value set-up, because they do not deform the initial hypersurface $\mathcal{N}$ in space-time. This shows that there is no loss of generality in assuming that (2.5) and (2.8) hold†.

It is natural to ask whether H defined by (2.6) can be removed by an appropriate choice of gauge. As shown by Bondi *et al.* (1962) in the smooth axisymmetric case, and as shown in § 4 in the general polyhomogeneous case, the condition

$$\lim_{r\to\infty} \beta = 0 \tag{2.9}$$

can always be achieved, at the price, however, of *deforming* $\mathcal{N}$ in space-time. Since it is our goal to analyse an initial value problem in which a null hypersurface $\mathcal{N}$ is given, we shall *not* assume that (2.9) holds unless explicitly specified otherwise.‡ (We will, however, find it useful to impose (2.9) when discussing the physical properties of the four-dimensional space-time. This is clearly justified by the results of § 4.)

Let us show that polyhomogeneity of h_{ab} implies that of β, U^a, V and $\partial h_{ab}/\partial u$:

Proposition 2.1. *Given any sequence* $\{N_i\}_{i=0}^{\infty}$, $N_0 = 0$, *there exists a sequence* $\{\tilde N_i\}_{i=0}^{\infty}$ *with* $\tilde N_0 = 0$, $\tilde N_1 = N_1$, $\tilde N_i \geqslant N_i$, *such that for all* $h_{ab} \in \mathcal{A}^{\{\tilde N_i\}} \cap C^0(\bar{\mathcal{N}})$ *satisfying* $\lim_{r\to\infty}\ h_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$ *we have*

(i)

$$\beta, U^a, r^{-2}V \in \mathcal{A}_{phg} \cap C^0(\bar{\mathcal{N}})\,; \tag{2.10}$$

(ii) If moreover

$$\lim_{r\to\infty} U^a = 0 \tag{2.11}$$

holds, then we have, for any $j \geqslant 0$,

$$r^{-1}V \in \mathcal{A}_{phg} \cap C^0(\bar{\mathcal{N}}), \tag{2.12}$$

$$\left(\frac{\partial}{\partial u}\right)^j h_{ab} \in \mathcal{A}^{\{\tilde N_i\}} \cap C^0(\bar{\mathcal{N}}). \tag{2.13}$$

Remarks. 1. It is rather clear from the gauge character of $\lim_{r\to\infty} U^a$ that (2.11) is not necessary for (2.13) to hold, with possibly a different sequence $\{\tilde N_i\}$. We have indeed verified this explicitly in the axisymmetric case (the validity of (2.13) is in this case guaranteed by a cancellation of some 'dangerous terms' which occur in the equation for $\partial h_{ab}/\partial u$). It should be noted that, assuming the initial value

† More precisely, there is no loss of generality in assuming that (2.5) holds, and there is no loss of generality in assuming that (2.8) holds provided the time-development of the data is sufficiently regular asymptotically in some neighbourhood of $\mathcal{N}$ (e.g. polyhomogeneous) to be able to perform the construction which leads to (2.8).

‡ Cf. also Hogan & Trautman (1987) and Hogan (1985) for an analysis in which (2.9?) is not assumed to hold, motivated by rather different considerations. The function e^H here corresponds precisely to the function $p_0 = \lim_{r\to\infty}\ p$ of Hogan & Trautman (1987).

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problem for polyhomogeneous data on $\bar{\mathcal{N}}$ can be solved, the u-dependent terms arising when $\lim_{r\to\infty} U^a \neq 0$ would lead to a u-dependent limit of h_{ab} differing from the round metric on surfaces other than $\bar{\mathcal{N}}$. This can also be seen from inspection of the transformation used above to remove X^a. It does not affect the Proposition above which applies only on $\bar{\mathcal{N}}$.

2. Assuming the characteristic initial value problem for polyhomogeneous data can be solved in the space of space-times with a polyhomogeneous $\mathscr{I}$, one way of understanding the significance of the relation between the $\{N_i\}_{i=0}^{\infty}$ and $\{\tilde{N}_i\}_{i=0}^{\infty}$ sequences is that if we are given initial data in a space characterized by the sequence $\{N_i\}_{i=0}^{\infty}$, then the sequence $\{\tilde{N}_i\}_{i=0}^{\infty}$ can be chosen to be the one appropriate to the evolution of that initial data. Sequences $\{\tilde{N}_i\}_{i=0}^{\infty}$ characterize those spaces which are invariant under evolution governed by the vacuum Einstein equations.

Proof. Replacing h_{ab} by

$$\frac{\sin\theta}{\sqrt{\det h_{ab}}}\, h_{ab}$$

and r by

$$r\frac{\sqrt{\det h_{ab}}}{\sin\theta}$$

we may without loss of generality assume $\sqrt{\det h_{ab}} = \sin\theta$. A simple but somewhat tedious analysis of the equations of Appendix C, making use of Proposition A.1 in Appendix A, gives the following: the limits $\lim_{r\to\infty}\beta$, $\lim_{r\to\infty} U^a$, exist and are, respectively, a smooth function and a smooth vector field on a sphere. We define

$$H \equiv \lim_{r\to\infty}\beta, \qquad X^a \equiv -\lim_{r\to\infty} U^a,$$

$$\psi^a \equiv 2\mathrm{e}^{2H}\mathcal{D}^a H, \tag{2.14}$$

where $\mathcal{D}^a$ is the covariant derivative with respect to the metric

$$\hat{h} = \lim_{r\to\infty} h_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b \quad \text{on} \quad S^2.$$

We then have

$$\beta - H \in \mathcal{A}_{phg}/r^2, \tag{2.15}$$

$$U^a + X^a - \psi^a/r \in \mathcal{A}_{phg}/r^2, \tag{2.16}$$

$$r^{-2}V \in \mathcal{A}_{phg} \cap C^0(\bar{\mathcal{N}}). \tag{2.17}$$

Let us also define

$$\psi \equiv \mathrm{e}^{2H}\left(1 + 2\Delta_{\hat{h}} H + 4|\mathcal{D}H|^2_{\hat{h}}\right), \tag{2.18}$$

where $\Delta_{\hat{h}}$ is the Laplacian of the metric $\hat{h}$, and $|\cdot|_{\hat{h}}$ denotes the norm in $\hat{h}$. If moreover $\lim_{r\to\infty} U^a = 0$, it follows that

$$V - \psi r \in \mathcal{A}_{phg}, \tag{2.19}$$

$$\frac{\partial h_{ab}}{\partial u} \in \frac{C^\infty(S^2)}{r} + \frac{\mathcal{A}_{phg}}{r^2}, \tag{2.20}$$

In the above analysis the only not entirely trivial step is to prove (2.20). Indeed, equations (C 5) and (C 6) take the form

$$\frac{\partial \phi_1}{\partial r} + f\phi_2 = \zeta_1, \tag{2.21}$$

$$\frac{\partial \phi_2}{\partial r} - f\phi_1 = \zeta_2, \tag{2.22}$$

where, using the notation of van der Burg (1966) and Appendix C,

$$\phi_1 = r\cosh(2\delta)\,\frac{\partial\gamma}{\partial u}, \quad \phi_2 = r\,\frac{\partial\delta}{\partial u}, \quad f = 2\sinh(2\delta)\,\frac{\partial\gamma}{\partial r}, \tag{2.23}$$

and the ζ_a, $a = 1,2$, can be found in Appendix C. The hypothesis $h_{ab} \in \mathcal{A}^{phg} \cap C^0(\bar{\mathcal{N}})$ is equivalent to $\gamma, \delta \in \mathcal{A}^{phg} \cap C^0(\bar{\mathcal{N}})$, so that from (2.15)–(2.18) one finds

$$rf,\ \zeta_a \in \mathcal{A}_{phg}/r^2, \qquad a = 1,2. \tag{2.24}$$

It is an exercise in ODEs to show that under (2.24) any solution of (2.21)–(2.22) is necessarily polyhomogeneous, with

$$\phi_a \in C^\infty(S^2) + r^{-1}\,\mathcal{A}_{phg}.$$

(The result can be proved by, for example, setting up a contraction principle argument† in weighted spaces of the kind used in Appendix B.) This implies

$$\frac{\partial h_{ab}}{\partial u} \in r^{-1}\,C^\infty(S^2) + r^{-2}\,\mathcal{A}_{phg},$$

as claimed.

To prove our claim about existence of *self-consistent* sequences $\{\tilde N_i\}$ which have the property that $h_{ab} \in \mathcal{A}^{\{\tilde N_i\}}$ is *formally* preserved under time evolution, define

$$N = N_1,$$

so that

$$|h_{ab} - \hat h_{ab}| = O(r^{-1}\log^N r).$$

Consider a term, say χ, in h_{ab} which has a radial behaviour $r^{-i}\log^j r$. From the equations of Appendix C one easily finds that such a term produces terms $r^{-i-1}\log^{N+j} r$ + lower order‡ in β. Next, if $i = 2$, such a term will produce terms $r^{-3}\log^{j+1} r$ + lower order in U^a, while for $i \neq 2$ it will lead to terms $r^{-i-1}\log^j r$ + lower order in U^a. Using the Kronecker δ^b_a defined as usual by

$$a, b \in \mathbb{R}\,, \qquad \delta^b_a = \begin{cases} 1, & \text{for } a = b,\\ 0, & \text{otherwise,}\end{cases}$$

we conclude that χ produces terms $r^{-i-1}\log^{j+\delta^2_i} r$ + lower order in U^a. There is a cancellation in the equation for V which implies that if $i = 1$ and N $(= j) = 0$, χ will generate terms $r^{-1}\log^{\hat N_1} r$ + lower order in V, rather than the $\log r$ + lower order terms which would have appeared if the cancellation had not

† It has been pointed out to us by R. Bartnik that the result can be easily established by replacing (2.21)–(2.22) by a single complex equation for the function $\phi_1 + \sqrt{-1}\,\phi_2$.

‡ By 'lower order' we mean terms which have the same power of r^{-1} and smaller powers of $\log r$, or higher powers of r^{-1}.

taken place: here $\hat{N}_1$ is determined by N_2 ($\hat{N}_1 = 0$ if $N_2 = 0$). In general, the cancellation implies that the leading term in V arising from $i = 1$ and $N > 0$ is of order $\log^N r$, while the leading contribution¶ from χ is of order $r^{1-i} \log^{j+\delta_i^2} r$. Inserting all this information in the equations for $\partial h_{ab}/\partial u$ one in general obtains a contribution $r^{-1-i} \log^{j+\delta_i^2} r$ + lower order from χ, but an $i = 1$, $N > 0$ term contributes‖ only a term of order $r^{-2} \log^{N-1} r$.

To construct a sequence $\{\tilde{N}_i\}$ given a sequence $\{N_i\}$, set $\tilde{N}_1 = N_1$. Then from the equations for $\partial h_{ab}/\partial u$ one obtains, if $N_1 > 0$,

$$\frac{\partial}{\partial u}\left(h_{ab} - \frac{h^1_{ab}(u, x^a)}{r}\right) = O\left(r^{-2} \log^{N_1 - 1} r\right), \tag{2.25}$$

for some $h^1_{ab}(u, x^a)$. If $N_1 = 0$, the next contributions to the u-derivatives of the γ and δ used in Appendix C are $O(r^{-3} \log^{N_2+1} r)$, which implies that the time derivative of the trace-free part of $h_{ab} - h^1_{ab}(u, x^a)/r$ is $O(r^{-3} \log^{N_2+1} r)$; on the other hand, the time derivative of the 'trace part' of $h_{ab} - h^1_{ab}(u, x^a)/r$ is $O(r^{-2})$ and is determined uniquely by the u-derivatives of $h^1_{ab}(u, x^a)$. (2.25) shows that the coefficients of the $r^{-1} \log^i r$, $i = 1, \ldots, N_1$ are constants of motion, and if we set $\tilde{N}_2 = \max(N_2, N_1 - 1)$ then the space $\mathcal{A}^{\{\tilde{N}_i\}}$ will be formally preserved by evolution up to $O(r^{-3+\epsilon})$, $\epsilon > 0$, terms. Proceeding recursively one can construct a self-consistent sequence $\{\tilde{N}_i\}$. The analysis of the higher u-derivatives proceeds in a similar manner by considering the equations obtained from the Einstein equations by u-differentiation, and the result follows. ■

From what has been said in the proof above it should be clear that if we write, along $\mathcal{N}$,

$$h_{ab} \sim \sum_{i=0}^{\infty} \sum_{j=0}^{N_i} h_{ijab}(\theta, \varphi)\, r^{-i} \log^j r,$$

then on $\mathcal{N}$ from the Einstein equations one obtains a Bondi–van-der-Burg–Metzner type hierarchy of equations

$$\frac{\partial h_{ijab}}{\partial u} = F_{ijab},$$

where F_{ijab} is a function of θ, φ and the $h_{k\ell cd}$, $0 \leqslant k \leqslant i-1$ together with a finite number of their derivatives. In particular, if we assume that a polyhomogeneous expansion of the metric also holds in a neighbourhood of $\mathcal{N}$ one obtains:

Proposition 2.2. *Under the hypotheses of Proposition 2.1, including (2.11), we have:*

(i) The coefficients of $r^{-1} \log^j r$, $1 \leqslant j \leqslant N_1$, in h_{ab} are constants of motion.

¶ The radial behaviour $r^{1-i} \log^{j+\delta_i^2} r$ is obtained here by a SHEEP calculation. It seems that knowledge of some cancellations for $i = 1$ is necessary for the argument to hold. Nevertheless for $i \geqslant 1$ a straightforward analysis of the V equation (C4) yields a leading order contribution $r^{1-i} \log^{j+\delta_i^1+\delta_i^2} r$ to V from χ, without going into the details of the cancellation structure of the equations (which SHEEP automatically does). One could use this estimate of the contributions of χ to V in the remainder of the argument to prove Proposition 2.1, with a perhaps somewhat 'worse' sequence $\tilde{N}_i$.

‖ This exponent is, again, obtained using SHEEP. Analytically it is more or less straightforward to estimate the right-hand side of (2.25) as $O(r^{-2} \log^{N_1+1} r)$. Such an estimate would be sufficient for the main conclusion to remain valid.

(ii) Suppose that $N_1 = 0$. Then the coefficients of $r^{-2}\log^j r$, $1 \leqslant j \leqslant \tilde{N}_2 = N_2$ in h_{ab} and the coefficients of r^{-2} in the trace-free part of the h_{ab} are constants of motion.

(iii) Suppose that on $\mathcal{N}$ we have

$$h_{ab} - \lim_{r\to\infty} h_{ab} = \frac{h^1_{ab}(x^a)}{r} + \frac{a(x^a)\breve{h}_{ab}(x^a)}{r^2} + O\left(r^{-3}\log^{N_3} r\right), \tag{2.26}$$

with $\breve{h}_{ab}$ given by (2.4), and that $N_1 = \ldots = N_i = 0$, $i \geqslant 2$. Then we can set $N_j = \tilde{N}_j$, $j = 0, \ldots, i+1$ and the coefficients of $r^{-i-1}\log^j r$, $1 \leqslant j \leqslant N_{i+1} = \tilde{N}_{i+1}$ in h_{ab} are constants of motion.

Remarks. In the notation of van der Burg (1966), (2.26) is equivalent to

$$\gamma = c/r + O(r^{-3}\log^{N_3} r), \quad \delta = d/r + O(r^{-3}\log^{N_3} r).$$

Note that the time dependence of the r^{-1} contributions to h_{ab} is undetermined by the above considerations, which are purely asymptotic†. In § 4 we show that the only remaining coordinate freedom is given by the BMS group, which implies that we have only one arbitrary function of θ and ϕ available for changing the values of the constants of motion just obtained, so that not more than one of them (if any) can be set to a fixed value.

It is natural to consider extending the second result in the Proposition and ask what is the 'smallest' self-consistent sequence $\tilde{N}_i$ if $N_i = 0$ for all i and it is not assumed *a priori* that (2.26) holds. When $\lim_{r\to\infty} U^a = 0$, one easily obtains, from what has been said earlier, that we can set

$$\tilde{N}_0 = \tilde{N}_1 = \tilde{N}_2 = 0, \quad \tilde{N}_3 = 1.$$

Moreover in such a case the arguments of the proof of Proposition 2.1 show that we will have

$$\begin{array}{lll} \beta \in \mathcal{A}^{\{N^\beta_i\}}, & N^\beta_0 = N^\beta_1 = N^\beta_2 = N^\beta_3 = 0, & N^\beta_4 = 1, \\ U^a \in \mathcal{A}^{\{N^U_i\}}, & N^U_0 = N^U_1 = N^U_2 = 0, & N^U_3 = 1, \\ V - r \in \mathcal{A}^{\{N^V_i\}}, & N^V_0 = 0, & N^V_1 = 1. \end{array}$$

In asymptotically Minkowskian coordinates

$$(t, x, y, z) = (u + r,\ r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta);$$

this corresponds to a metric (2.1) which along $\mathcal{N}$ approaches the Minkowski one as

$$\gamma_{\mu\nu} - \eta_{\mu\nu} = \frac{\gamma^1_{\mu\nu}}{r} + \frac{\gamma^{2,1}_{\mu\nu}\log r}{r^2} + \frac{\gamma^2_{\mu\nu}}{r^2} + O(r^{-2-\epsilon}), \quad \epsilon > 0.$$

The results of Proposition 2.1 can be generalized to include both matter fields and rather weaker asymptotic conditions. Let us start by defining a space of functions $C^{\mu,\lambda}_\infty$: for $\mu, \lambda \in \mathbb{R}$ a function f will be said to be in $C^{\mu,\lambda}_\infty(\mathcal{N})$, or for

† These contributions define an analogue of what is called a 'news function' in Bondi *et al.* (1962) and Sachs (1962), and should be determined by the behaviour of sources and/or of the gravitational field in the interior, or perhaps by some interior boundary conditions, in a complete solution.

short in $C^{\mu,\lambda}_{\infty}$, if for all $i \in \mathbb{N}$ and for all multi-indices α we have

$$|\partial^i_r \partial^\alpha_v f| \leqslant C_{i,\alpha} r^{-\mu-i}(1+|\log r|)^\lambda,$$

for some constants $C_{i,\alpha}$, where the coordinates v stand for θ, ϕ. In order to avoid in what follows a rather annoying discussion of some not so interesting special cases we shall always assume that in all spaces considered if $\mu = 1, 2, 3$ or 4 then the logarithmic behaviour exponent λ satisfies $\lambda \neq -2, -1$. We wish to show that the space of metrics on $\mathcal{N}$ of the form

$$h_{ab} - \breve{h}_{ab} \in r^{-1}\mathcal{A}^{phg} + C^{\mu,\lambda}_{\infty}, \qquad \mu > 0 \tag{2.27}$$

is *formally* preserved by evolution with the Einstein equations with possibly some matter fields. Let us denote by $\{\text{l.h.s.}\}_n$, respectively by $\{\text{r.h.s.}\}_n$, the left-hand side, respectively the right-hand side, of the nth equation of Appendix C. Then, in the presence of matter the Einstein equations become

$$\{\text{l.h.s.}\}_1 = \{\text{r.h.s.}\}_1 + \tfrac{1}{4} r \hat{T}_{11}, \tag{2.28}$$

$$\{\text{l.h.s.}\}_2 = \{\text{r.h.s.}\}_2 + 2r^2 \hat{T}_{12} \operatorname{cosec}\theta, \tag{2.29}$$

$$\{\text{l.h.s.}\}_3 = \{\text{r.h.s.}\}_3 + 2r^2 \hat{T}_{13}, \tag{2.30}$$

$$\frac{\partial V}{\partial r} = \{\text{l.h.s.}\}_4 = \{\text{r.h.s.}\}_4 - \tfrac{1}{2} e^{2\beta} h^{ab} \hat{T}_{ab}, \tag{2.31}$$

$$\{\text{l.h.s.}\}_5 = \{\text{r.h.s.}\}_5 + \frac{e^{2\beta}}{4r}(e^{-2\gamma}\hat{T}_{22} - e^{2\gamma}\sin^{-2}\theta\, \hat{T}_{33}), \tag{2.32}$$

$$\{\text{l.h.s.}\}_6 = \{\text{r.h.s.}\}_6 + \frac{e^{2\beta}}{2r\cosh 2\delta \sin\theta}(\hat{T}_{23} - \tfrac{1}{2} h^{cd}\hat{T}_{cd} h_{23}), \tag{2.33}$$

where h^{ab} is the matrix inverse to h_{ab}, $\hat{T}_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2} T\gamma_{\mu\nu})$, κ is the gravitational coupling constant, and $T_{\mu\nu}$ is the energy momentum tensor of the matter fields. We shall require

$$\hat{T}_{11} \in r^{-3}\mathcal{A}^{phg} + C^{\mu_{11},\lambda_{11}}_{\infty}, \qquad \mu_{11} > 2, \tag{2.34}$$

$$\hat{T}_{1a} \in r^{-2}\mathcal{A}^{phg} + C^{\mu_{1a},\lambda_{1a}}_{\infty}, \quad a = 2,3, \quad \mu_{12} = \mu_{13} > 1, \quad \lambda_{12} = \lambda_{13}, \tag{2.35}$$

$$h^{ab}\hat{T}_{ab} \in r^{-1}\mathcal{A}^{phg} + C^{\mu_0,\lambda_0}_{\infty}, \qquad \mu_0 > 0, \tag{2.36}$$

$$e^{2\gamma}\sin^{-2}\theta\, \hat{T}_{33} - e^{-2\gamma}\hat{T}_{22} \in r^{-1}\mathcal{A}^{phg} + C^{\mu_r,\lambda_r}_{\infty}, \qquad \mu_r > 0, \tag{2.37}$$

$$\hat{T}_{23} - \tfrac{1}{2} h^{cd}\hat{T}_{cd} h_{23} \in r^{-1}\mathcal{A}^{phg} + C^{\mu_r,\lambda_r}_{\infty}, \tag{2.38}$$

where the various powers in front of $\mathcal{A}^{phg}$ and the exponents in $C^{\mu_*,\lambda_*}_{\infty}$ have been chosen so that the leading order behaviour of the various functions which appear in the metric coincides with the behaviour one observes in the vacuum case. Assuming that $\lim_{r\to\infty}\beta = \lim_{r\to\infty} U^a = 0$, an analysis as described in the proof of Proposition 2.1 leads to

$$\frac{\partial}{\partial u} h_{ab} \in r^{-1}\mathcal{A}^{phg} + C^{\mu+1,\lambda+\delta^2_\mu}_{\infty} + C^{\mu_{11}-1,\lambda_{11}+\delta^4_{\mu_{11}}}_{\infty}$$
$$+ C^{\mu_{1a},\lambda_{1a}+\delta^3_{\mu_{1a}}}_{\infty} + C^{\mu_r+1,\lambda_r}_{\infty} + C^{\mu_0+2,\lambda_0+\delta^1_{\mu_0}+N_1}_{\infty}. \tag{2.39}$$

It follows that for

$$\mu \leqslant \min\{\mu_{11}-1, \mu_{1a}, \mu_r+1, \mu_0+2\}$$

(with appropriate inequalities for the λ_*s if the above inequality is an equality) the Einstein equations will *formally* preserve the space of metrics h_{ab} satisfying (2.27). The proof of (2.39) follows immediately by integration of the equations (2.28)–(2.33), and of the equations which are obtained by u-differentiation of those. Step by step one obtains:

$$\begin{aligned}
\beta &\in r^{-1}\mathcal{A}^{phg} + C_\infty^{\mu+1,\lambda+N_1} + C_\infty^{\mu_{11}-2,\lambda_{11}},\\
U^a &\in r^{-2}\mathcal{A}^{phg} + C_\infty^{\mu+1,\lambda+\delta^2_\mu} + C_\infty^{\mu_{11}-1,\lambda_{11}+\delta^4_{\mu_{11}}} + C_\infty^{\mu_{1a},\lambda_{1a}+\delta^3_{\mu_{1a}}},\\
V-r &\in \quad \mathcal{A}^{phg} + C_\infty^{\mu-1,\lambda+\delta^1_\mu+\delta^2_\mu} + C_\infty^{\mu_0-1,\lambda_0+\delta^1_{\mu_0}}\\
&\qquad + C_\infty^{\mu_{11}-3,\lambda_{11}+\delta^3_{\mu_{11}}+\delta^4_{\mu_{11}}} + C_\infty^{\mu_{1a}-2,\lambda_{1a}+\delta^2_{\mu_{1a}}+\delta^3_{\mu_{1a}}};
\end{aligned}$$

the above equations inserted in the evolution equations (2.32)–(2.33) yield (2.39). (If we assume that the appropriate decay properties are also satisfied by $\partial^j\hat{T}_{\mu\nu}/\partial u^j$ on $\mathcal{N}$, $j=0,\ldots,J$, then a preliminary examination suggests that $\partial^i h_{ab}/\partial u^i$ will also be of the form (2.39) for $i=0,\ldots,J$.) We note that the previous calculations on similar lines (Couch & Torrence 1972; Novak & Goldberg 1982) led to upper bounds on the parameter μ which do not appear here because, unlike those previous calculations, we do not forbid the appearance of $\log r$ terms.

3. Geometric interpretation

Consider a metric of the form (2.1). Following Penrose (1965) it is useful to introduce a new coordinate

$$x \equiv r^{-1},$$

so that by Proposition 2.1 when $h_{ab}\in\mathcal{A}^{phg}\cap C^0(\bar{\mathcal{N}})$ the metric

$$\begin{aligned}
\tilde{\gamma}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu &\equiv x^2\,\gamma_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\\
&= -Vx^3\,\mathrm{e}^{2\beta}\,\mathrm{d}u^2 + 2\mathrm{e}^{2\beta}\,\mathrm{d}u\,\mathrm{d}x + h_{ab}(\mathrm{d}x^a+U^a\,\mathrm{d}u)(\mathrm{d}x^b+U^b\,\mathrm{d}u) \quad (3.1)
\end{aligned}$$

is *polyhomogeneous* on the set $\{x\in[0,1/R],\ x^a\in S^2\}$, i.e. there exists a sequence $\{\hat{N}_i\}$ and functions $\tilde{\gamma}_{\mu\nu ij}(x^a)\in C^\infty(S^2)$ such that

$$\tilde{\gamma}_{\mu\nu} \sim \sum_{i=0}^{\infty}\sum_{j=0}^{\hat{N}_i}\tilde{\gamma}_{\mu\nu ij}(x^a)\,x^i\,\log^j x, \qquad \hat{N}_0=0.$$

(When (2.26) holds and $h_{ab}\in C^\infty(\bar{\mathcal{N}})$ one actually obtains $\hat{N}_i=0$ for all i, and the metric (3.1) is in $C^\infty(\bar{\mathcal{N}})$. This corresponds to the standard Bondi–Penrose–Sachs situation of a smooth $\mathscr{I}$.) We have the following result, which is established by calculating the Christoffel symbols of the metric $x^2\gamma_{\mu\nu}$ using SHEEP, and making use of vacuum Einstein equations for $\gamma_{\mu\nu}$ in the first leading orders:

Proposition 3.1. *Consider a characteristic initial data set with* $h_{ab}\in\mathcal{A}_{phg}\cap C^0(\bar{\mathcal{N}})$, $\lim_{r\to\infty}h_{ab}=\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\varphi^2$ *and* $\lim_{r\to\infty}U^a=0$, *and set* $\Omega=x=r^{-1}$.

We have

$$\lim_{x\to 0} \tilde{\nabla}_a \tilde{\nabla}_b \Omega = 0, \qquad a, b = 2, 3, \tag{3.2}$$

where $\tilde{\nabla}$ *is the covariant derivative of the metric* $\tilde{\gamma}_{\mu\nu} \equiv x^2 \gamma_{\mu\nu}$. *If moreover* $\lim_{x\to 0} \beta = 0$, *then we also have*

$$\lim_{x\to 0} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \Omega = 0, \qquad \mu, \nu = 0, \dots, 3. \tag{3.3}$$

Recall that the geometric meaning of (3.2) is the vanishing of the shear of the hypersurface $\{x = 0\}$ (cf. Wald 1984). Another interpretation of (3.2) is that the *conformal extrinsic curvature* of $\mathscr{I}$ vanishes, cf. the Appendix of Andersson & Chruściel (1994*a*). This property of $\mathscr{I}$ is well known for $\tilde{\gamma}_{\mu\nu} \in C^\infty(\bar{\mathcal{N}})$. It should be stressed that for general polyhomogeneous metrics as constructed (on $\mathcal{N}$) in the previous section, the fact that the left-hand sides of (3.2)–(3.3) exist and are bounded is a non-trivial statement which *makes use* of vacuum Einstein equations to the first two leading orders, because equations (3.2)–(3.3) contain derivatives of the metric which could potentially blow up as $\log^{\hat{N}_1} x$ as $x \to 0$. Proposition 3.1 is the original observation which led to the proof in Andersson & Chruściel (1994*b*), that generic Cauchy data constructed by the conformal method as in Andersson & Chruściel (1994*a*) will lead to space-times which *cannot* admit a polyhomogeneous $\mathscr{I}$.

Throughout the remainder of this section we shall assume that

$$\lim_{r\to\infty} h_{ab} \mathrm{d}x^a \mathrm{d}x^b = \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2 =: \check{h}_{ab} \mathrm{d}x^a \mathrm{d}x^b, \tag{3.4}$$

$$\lim_{r\to\infty} \beta = 0, \qquad \lim_{r\to\infty} U^a = 0. \tag{3.5}$$

A textbook property of smooth $\mathscr{I}$s is that the Weyl tensor of the conformally rescaled metric vanishes at $\mathscr{I}$ (cf. Wald 1984). A SHEEP calculation of the Weyl tensor of the metric $\tilde{\gamma}_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu$, assuming that $\gamma_{\mu\nu}$ is vacuum, gives:

Proposition 3.2. *In addition to the hypotheses of Proposition 3.1, let* $h_{ab} \in C^2(\bar{\mathcal{N}})$, *and let (3.4)–(3.5) hold. In local coordinates define*

$$\chi_{ab} = \lim_{x\to 0} \left(\frac{\partial^2 h_{ab}}{\partial x^2} - \tfrac{1}{2} h^{cd} \frac{\partial^2 h_{cd}}{\partial x^2} \, h_{ab} \right).$$

Let $\tilde{C}_{\alpha\beta\gamma\delta}$ *denote the components of the Weyl tensor of* $\tilde{\gamma}$ *in the half-null tetrad*

$$\theta^0 = \mathrm{e}^{2\beta} \mathrm{d}u, \qquad \theta^1 = \tfrac{1}{2} x^3 V \mathrm{d}u - \mathrm{d}x,$$

$$\theta^2 = -(\cosh\delta \, \mathrm{e}^\gamma \, U^\theta + \sin\theta \, \sinh\delta \, \mathrm{e}^{-\gamma} U^\phi) \, \mathrm{d}u + \cosh\delta \, \mathrm{e}^\gamma \, \mathrm{d}\theta + \sin\theta \, \sinh\delta \, \mathrm{e}^{-\gamma} \, \mathrm{d}\varphi,$$

$$\theta^3 = -(\sinh\delta \, \mathrm{e}^\gamma \, U^\theta + \sin\theta \, \cosh\delta \, \mathrm{e}^{-\gamma} U^\phi) \, \mathrm{d}u + \sinh\delta \, \mathrm{e}^\gamma \mathrm{d}\theta + \sin\theta \, \cosh\delta \, \mathrm{e}^{-\gamma} \, \mathrm{d}\varphi. \tag{3.6}$$

Then we have

$$\lim_{x\to 0} \tilde{C}_{1a1b} = \chi_{ab}, \tag{3.7}$$

while all the remaining components of $\lim_{x\to 0} \tilde{C}_{\alpha\beta\gamma\delta}$ *vanish, except of course those which can be obtained by appropriate permutations of indices of* $\tilde{C}_{1a1b}$.

(Let us point out that Proposition 3.2 will still be true with non-vanishing Ricci curvature provided that (2.27) and (2.34)–(2.38) hold with $\mu_0, \mu_r > 1$, $\mu_{1a} > 2$, $\mu, \mu_{11} > 3$, and that the powers of r^{-1} in front of the polyhomogeneous pieces in $\hat{T}_{\mu\nu}$ in (2.34)–(2.38) are increased by one.) Proposition 3.2 shows that the Weyl tensor of $\tilde{\gamma}_{\mu\nu}$ vanishes at $x = 0$ if and only if the trace free part of $\partial^2 h_{ab}/\partial x^2|_{x=0}$ vanishes, i.e. if and only if what Bondi *et al.* termed the 'outgoing radiation condition', (2.26), holds. Let us also mention that (3.7) is equivalent to

$$\tilde{\Psi}_0\big|_{x=0} = \tilde{\Psi}_1\big|_{x=0} = \tilde{\Psi}_2\big|_{x=0} = \tilde{\Psi}_3\big|_{x=0} = 0, \qquad \tilde{\Psi}_4\big|_{x=0} \neq 0,$$

where the $\tilde{\Psi}_i$s are the Newman–Penrose components of the Weyl tensor of the metric $\tilde{\gamma}$ in a null tetrad related to the tetrad (3.6) above in the obvious way.

It is worthwhile emphasizing that it follows from Proposition 2.2, point 2, and from (3.7) that the components $\tilde{C}_{1a1b}(u, 0, x^a)$ are *pointwise constants of motion*, i.e. independent of u. This result seems to have been already observed by Winicour (1985), and independently by Christodoulou & Klainerman (1993) (under much weaker asymptotic conditions).

Perhaps the most important result of the Bondi–Sachs analysis is the well known theorem (originally due to Trautman (1958)) that Bondi's mass is a decreasing function of u. For the metrics under consideration here, let us define the Bondi mass as $(-\frac{1}{2})$ the integral over the sphere S^2 of the r^0 coefficient in the expansion of V. This definition clearly reduces to the original one by Bondi–Sachs, when the conditions imposed by Bondi and Sachs hold. We have found that, under (3.4)–(3.5), and whatever the sequence $\{N_i\}$ and the $h_{ab} \in \mathcal{A}^{\{N_i\}}$, the mass loss eqn (35) of Bondi *et al.* (1962), which can be obtained by equating to zero the integral over S^2 of the right side of (3.8), (cf. also Sachs 1962, eqn (4.16)) remains unchanged. This can be seen as follows: under the above conditions it follows from the vacuum Einstein equations that

$$V - r,\ r^2\beta,\ r^2U^a \in \mathcal{A}^{phg}.$$

As has been shown by Sachs (1962) following the original observation of Bondi *et al.* (1962), in the coordinate system of (2.1) we have (if the other field equations hold)

$$R_{00} = 0 \quad \Leftrightarrow \quad \lim_{r\to\infty} r^2 R_{00} = 0.$$

A SHEEP calculation gives

$$\lim_{r\to\infty} r^2 R_{00} = \lim_{r\to\infty} \left(\frac{\partial V}{\partial u} - \tfrac{1}{4} r^4 h^{ab} h^{cd} \frac{\partial^2 h_{ac}}{\partial u \partial r} \frac{\partial^2 h_{bd}}{\partial u \partial r} - r^2 \mathcal{D}_a \frac{\partial U^a}{\partial u} \right), \tag{3.8}$$

where $\mathcal{D}_a$ is the covariant derivative of the metric $\breve{h}_{ab}$. It is clear from (3.8) and from what has been said before that those log terms which could contribute to this equation, if any, drop out because their u derivatives vanish. It follows that for *all polyhomogeneous* initial data the Bondi mass is a non-increasing function of time when† $\lim_{r\to\infty} \beta = 0$.

We would like to point out that it is clear that the mass as defined above is the 'correct mass' for polyhomogeneous initial data $h_{ab} \in \mathcal{A}^{\{N_i\}}$ with $N_0 = N_1 = 0$. On the other hand we believe that some care should be taken when interpreting

† When $\lim_{r\to\infty} \beta \neq 0$, then the 'mass aspect' is no longer given by the standard Bondi–Sachs expression. But monotonicity of an appropriately defined total Bondi mass of course remains valid.

the above as the mass in the case $N_1 \neq 0$, since in that case the leading order behaviour of $V-r$ is logarithmic, which might reflect an infinite or ill defined mass of the system. Such a possibility is also suggested by (2.31), which shows that some log terms might arise from $1/r$ terms in $h^{ab}\hat{T}_{ab}$. Now in an orthonormal tetrad in which e^0 is *timelike* one finds that $h^{ab}\hat{T}_{ab} = -\kappa\, r^2(T^0_0 + T^1_1) = \kappa\, r^2(T_{00} - T_{11})$, so that $1/r$ terms in $h^{ab}\hat{T}_{ab}$ correspond to an infinite amount of matter energy: if $r^2 T_{00}$ behaves as $1/r$, then T_{00} (the matter energy density) ceases to be integrable over $\mathcal{N}$. A thorough analysis of the conditions under which the mass at null infinity is finite and well defined lies outside the scope of this paper.

It is natural to ask what happens with the 'peeling-off' property for space-times for which (2.3) fails to hold. A SHEEP calculation shows that (in the vacuum case) for any $h_{ab} \in C^0(\bar{\mathcal{N}}) \cap \mathcal{A}^{phg}$ along $\mathcal{N}$ we have

$$R_{\alpha\beta\gamma\delta} = \frac{R^1_{\alpha\beta\gamma\delta}}{r} + \frac{R^2_{\alpha\beta\gamma\delta}}{r^2} + \frac{R^{\log}_{\alpha\beta\gamma\delta}\,\log r}{r^3} + \frac{R^3_{\alpha\beta\gamma\delta}}{r^3} + \cdots,$$

with $R^1_{\alpha\beta\gamma\delta}$ and $R^2_{\alpha\beta\gamma\delta}$ peeling off exactly in the same way as they would if (2.3) were satisfied, i.e. $R^1_{\alpha\beta\gamma\delta}$ is of type N, and $R^2_{\alpha\beta\gamma\delta}$ is of type III; in fact $R^1_{\alpha\beta\gamma\delta}$ and $R^2_{\alpha\beta\gamma\delta}$ are exactly the same as in the case considered by Sachs (1962) (cf. also Trautman 1958). This follows from the fact that *all* terms which contribute to $R^1_{\alpha\beta\gamma\delta}$ and $R^2_{\alpha\beta\gamma\delta}$ are u-differentiated, and those log terms which could potentially contribute at this order are constants of motion. Here $R^{\log}_{\alpha\beta\gamma\delta}$ may contain powers of $\log r$.

In the case $\tilde{\gamma}_{\mu\nu} \in C^\infty(\bar{\mathcal{N}})$ it was observed in Newman & Penrose (1965, 1968) that there exist some nontrivial global constants of motion for a vacuum gravitating system. We wish to point out that these quantities cease to be constants of motion even in the case in which $h_{ab} \in C^\infty(\bar{\mathcal{N}})$ if one does not assume that (2.26) holds. Clearly it is sufficient to prove that assertion for those metrics (2.1) which are of the Bondi–van der Burg–Metzner form (Bondi *et al.* 1962):

$$\frac{\partial \gamma_{\mu\nu}}{\partial \varphi} = 0, \tag{3.9}$$

$$h_{ab}\,dx^a\,dx^b = e^{2\gamma}\,d\theta^2 + e^{-2\gamma}\,\sin^2\theta\,d\varphi^2, \tag{3.10}$$

$$U^\varphi = 0. \tag{3.11}$$

Let us expand the function γ appearing in (3.10) as

$$\gamma = \frac{c}{r} + \frac{\gamma_2}{r^2} + \frac{\gamma_{3,1}\log r}{r^3} + \frac{\gamma_3}{r^3} + \frac{\gamma_{4,1}\log r}{r^4} + \frac{D}{r^4} + \cdots.$$

(The terms $\gamma_{3,1}$ and $\gamma_{4,1}$ above are necessary: even if $\gamma_{3,1}|_{u=0} = \gamma_{4,1}|_{u=0} = 0$ (which we are free to assume), we shall have $\gamma_{3,1} \neq 0 \neq \gamma_{4,1}$ at later times in general, as a consequence of the evolution equations.) In the axisymmetric case with $\gamma_2 = \gamma_{3,1} \equiv 0$ the constant of motion is given by (van der Burg 1966)

$$\mathcal{D} = \int_{S^2} D\sin^2\theta\,d\mu_0, \qquad d\mu_0 = \sin\theta\,d\theta\,d\phi.$$

If one takes the minimal sequence of $\tilde{N}_i$ as described in §2 and allows $\gamma_2 \not\equiv 0$, one finds that the $\gamma_{3,1}$ and $\gamma_{4,1}$ terms (but not $r^{-3}\log^j r$ or $r^{-4}\log^j r$ with $j > 1$) arise. After a long calculation it turns out that the u-derivative of the $\gamma_{4,1}$ term, when multiplied by $\sin^3\theta$, is a total derivative with respect to θ and, removing further total derivative terms, one finds that the equation of motion for $\mathcal{D}$ takes the form,

$$\frac{\partial \mathcal{D}}{\partial u} = \int_{S^2} G \,\mathrm{d}\theta\,\mathrm{d}\phi,$$

where

$$G = \{\gamma_2 \sin\theta [\, \sin^2\theta\,(-4\partial^2 c/\partial\theta^2 + 16M - 10u + 4c) \\ -24\sin\theta\cos\theta\,\partial c/\partial\theta - 16\cos^2\theta\, c] + 15\sin^3\theta\,\gamma^0_{3,1}\}/12, \quad (3.12)$$

$$\gamma^0_{3,1} \equiv \gamma_{3,1}\Big|_{u=0},$$

and M is the usual Bondi mass aspect (i.e. $-2M$ is the integration constant which appears when integrating the V equation (C 4)).

For given $M|_{u=0} \not\equiv 0$, $\gamma_{3,1}|_{u=0}$ (perhaps, but not necessarily, being zero) and $c|_{u=0}$ it seems obvious from (3.12) that the function $\gamma_2|_{u=0}$ can be chosen so that we have

$$\partial\mathcal{D}/\partial u|_{u=0} \neq 0.$$

It must be pointed out that the above argument falls short of being a rigorous proof: the function M has a global character, and in particular it *might not* be independent of $\gamma_2|_{u=0}$ and $\gamma_{3,1}|_{u=0}$. (Nevertheless the above calculation shows that no obvious miraculous cancellations occur, and we find it completely implausible that in, say, the vacuum case there exists some kind of conspiracy between the functions $\gamma_2|_{u=0}$, $\gamma_{3,1}|_{u=0}$, c and M which leads to the *identical* vanishing of the θ-integral of G.)

It is curious, and not entirely unexpected, that in the axisymmetric polyhomogeneous setting there is a Newman–Penrose type quantity which is again a constant of motion. Define

$$\mathcal{Q} = \int_{S^2} \gamma_{4,1} \sin^2\theta \,\mathrm{d}\mu_0 \,. \quad (3.13)$$

As we show in Appendix D, $\mathcal{Q}$ is conserved by the evolution via the vacuum Einstein equations. It seems clear to us that an analogous result will be true in the general case, without assuming axisymmetry.

4. Existence of Bondi coordinates

Consider a metric γ defined on the set

$$\mathcal{U}_{\hat{R},\hat{C}_1,\hat{C}_2} \equiv \{\hat{r} \geqslant \hat{R},\ \hat{u} \in (\hat{C}_1, \hat{C}_2),\ \hat{x}^a \in S^2\}, \quad (4.1)$$

and suppose that there exists $0 < \epsilon < 1$ such that

$$\text{for}\quad (\hat{x}^\mu \hat{x}^\nu) = (\hat{u}\hat{u}), (\hat{u}\hat{x}^a), (\hat{r}\hat{x}^a) \qquad \gamma_{\hat{x}^\mu\hat{x}^\nu} \in \mathcal{A}_{phg}, \qquad |\gamma_{\hat{x}^\mu\hat{x}^\nu}| \leqslant \epsilon^{-1}, \quad (4.2)$$

$$\gamma_{\hat{r}\hat{u}} \in \mathcal{A}_{phg}, \qquad \epsilon \leqslant \gamma_{\hat{r}\hat{u}} \leqslant \epsilon^{-1}, \quad (4.3)$$

$$\gamma_{\hat{r}\hat{r}} \in \hat{r}^{-2}\mathcal{A}_{phg}, \qquad |\hat{r}^2\gamma_{\hat{r}\hat{r}}| \leqslant \epsilon^{-1}, \quad (4.4)$$

$$\gamma_{\hat{x}^b\hat{x}^a} \in \hat{r}^2\mathcal{A}_{phg}, \qquad |\hat{r}^{-2}\gamma_{\hat{x}^b\hat{x}^a}| \leqslant \epsilon^{-1}. \quad (4.5)$$

Following Penrose (1965) let $\hat{x} := \hat{r}^{-1}$, and set

$$\tilde{\gamma}_{\mu\nu} \equiv \hat{x}^2 \gamma_{\mu\nu}.$$

From (4.1)–(4.5) it follows that the metric $\tilde{\gamma}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$ can be extended by continuity to a polyhomogeneous metric on the set

$$\mathcal{V}_{1/\hat{R},\hat{C}_1,\hat{C}_2} \equiv \{0 \leqslant \hat{x} \leqslant 1/\hat{R},\ \hat{u} \in (\hat{C}_1, \hat{C}_2),\ \hat{x}^a \in S^2\}\ . \tag{4.6}$$

Actually the conditions (4.3)–(4.5) guarantee only that the appropriately rescaled functions $\tilde{\gamma}_{\mu\nu}$ can be extended by continuity to the boundary, with the appropriately rescaled metric degenerating perhaps at the boundary. We thus add the supplementary restriction that $\tilde{\gamma}_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b$ is *non-degenerate up-to-boundary*, and that $\tilde{\gamma}$ is also *non-degenerate with signature* $(-+++)$ *up-to-boundary*, in a sense which should be clear from what is said below. Define

$$\mathscr{I} = \{\hat{x} = 0,\ \hat{u} \in (\hat{C}_1, \hat{C}_2),\ \hat{x}^a \in S^2\}.$$

Throughout this paper we shall suppose that $\mathscr{I}$ is a null hypersurface; as in the smooth case (cf. Wald 1984), in the polyhomogeneous case this will necessarily hold if γ is vacuum. Let $\breve{u}$ be any smooth function on $\mathscr{I}$, and extend $\breve{u}$ to a smooth function defined in some neighbourhood of $\mathscr{I}$ in any way. Let $w_\mu\,\mathrm{d}x^\mu$ be any smooth nowhere vanishing one-form field defined in a neighbourhood of $\mathscr{I}$ such that $w_\mu X^\mu = 0$ for all $X^\mu \in T\mathscr{I}$; from the fact that $\mathscr{I}$ is null it is easily seen that

$$(\tilde{\gamma}^{\mu\nu} w_\mu w_\nu)|_{\mathscr{I}} = 0. \tag{4.7}$$

On $\mathscr{I}$ consider the one-form field

$$k_\mu|_{\mathscr{I}}\,\mathrm{d}x^\mu \equiv a\,w_\mu|_{\mathscr{I}}\,\mathrm{d}x^\mu + \mathrm{d}\breve{u} \tag{4.8}$$

with some function a; from (4.7) it follows that the equation

$$(\tilde{\gamma}^{\mu\nu} k_\mu k_\nu)|_{\mathscr{I}} = (2a\tilde{\gamma}^{\mu\nu} w_\mu \breve{u}_\nu + \tilde{\gamma}^{\mu\nu}\breve{u}_\mu\breve{u}_\nu)|_{\mathscr{I}} = 0 \tag{4.9}$$

$(\breve{u}_\mu \equiv \partial\breve{u}/\partial x^\mu)$ will have a (unique) smooth solution $a|_{\mathscr{I}}$ provided that

$$(\tilde{\gamma}^{\mu\nu} w_\mu \breve{u}_\nu)|_{\mathscr{I}} \quad \text{is bounded away from zero,} \tag{4.10}$$

which we shall assume to hold. (Note that (4.10) implies that $k^\mu \equiv \tilde{\gamma}^{\mu\nu}k_\nu$ will be transverse to $\mathscr{I}$.) By Proposition B.1 point 1 for every $(\breve{u}, \hat{x}^a) \in \mathscr{I}$ there exists a null geodesic $x^\mu(s, \breve{u}, \hat{x}^a)$ such that $\mathrm{d}x^\mu/\mathrm{d}s(0, \breve{u}, \hat{x}^a) = k^\mu|_{\mathscr{I}}$. There also exists a diffeomorphism $\hat{x}^a \to \breve{x}^a(\hat{x}^b)$ such that in the coordinates $(\breve{u}, \breve{x}^a)$ we have at $\breve{u} = 0$

$$\tilde{\gamma}_{ab}|_{u=x=0}\mathrm{d}\breve{x}^a\mathrm{d}\breve{x}^b = \phi^2 \breve{h}_{ab}\mathrm{d}\breve{x}^a\mathrm{d}\breve{x}^b \equiv \phi^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2),$$

where $\phi \in C^\infty(S^2)$ is uniformly bounded away from zero. Let (u, s, x^a) be obtained by Lie dragging $(\breve{u}, \breve{x}^a)$ along the integral curves $\hat{x}^\mu(s, \breve{u}, \hat{x}^a)$. Thus if we set $k^\mu \equiv \partial\hat{x}^\mu/\partial s$, then

$$k^\mu u_{,\mu} = 0, \qquad u(0, \breve{u}, \breve{x}^a) = \breve{u}, \tag{4.11}$$

$$k^\mu x^a{}_{,\mu} = 0, \qquad x^a(0, \breve{u}, \breve{x}^b) = \breve{x}^a. \tag{4.12}$$

By point 2 of Proposition B.1 and by the implicit function theorem there exists a neighbourhood of $\mathscr{I}$ on which (u, s, x^a) form a coordinate system. As k^μ is tangent to null geodesics we have $\tilde{\gamma}_{\mu\nu}k^\mu k^\nu = 0$, so that

$$\tilde{\gamma}(\partial/\partial s, \partial/\partial s) = 0.$$

It follows that in these coordinates $\tilde{\gamma}$ takes the form

$$\tilde{\gamma}_{\mu\nu}\mathrm{d}\hat{x}^\mu \mathrm{d}\hat{x}^\nu = \tilde{\gamma}_{uu}\mathrm{d}u^2 + 2\tilde{\gamma}_{us}\mathrm{d}u\,\mathrm{d}s + \tilde{\gamma}_{ab}(\mathrm{d}x^a + U^a\mathrm{d}u)(\mathrm{d}x^b + U^b\mathrm{d}u). \tag{4.13}$$

Setting $\check{r} = 1/s$ the original metric γ takes the form

$$\gamma_{\mu\nu}\,\mathrm{d}\hat{x}^\mu\,\mathrm{d}\hat{x}^\nu = \hat{r}^2\tilde{\gamma}_{uu}\,\mathrm{d}u^2 - 2(\hat{r}/\check{r})^2\tilde{\gamma}_{us}\,\mathrm{d}u\mathrm{d}\check{r} + \hat{r}^2\tilde{\gamma}_{ab}(\mathrm{d}x^a + U^a\,\mathrm{d}u)(\mathrm{d}x^b + U^b\,\mathrm{d}u),$$

where all functions in $\gamma_{\mu\nu}$ are polyhomogeneous, after appropriate rescalings. Finally define

$$r \equiv \left[\sin^2\theta\,\det\left(\gamma^{\mu\nu}\frac{\partial x^a}{\partial\hat{x}^\mu}\frac{\partial x^b}{\partial\hat{x}^\nu}\right)\right]^{-1/4}. \tag{4.14}$$

One easily finds

$$\partial r/\partial\hat{r} = \phi + O(\log^N\check{r}/\check{r})$$

for some N, so that there exist constants R, C_1, C_2 such that (r, u, x^a) as constructed above form a coordinate system on $\mathcal{U}_{R,C_1,C_2}$, where $\mathcal{U}_{R,C_1,C_2}$ is given by (4.1) (in the coordinates (r, u, x^a)). Going to this coordinate system from (4.14) one concludes that

$$\det(\gamma_{ab}) = r^4\sin^2\theta.$$

In this way one obtains a metric of the form (2.1), which satisfies appropriate Bondi requirements at $u = 0$. If one moreover assumes that γ is vacuum, then the Einstein equations imply

$$\frac{\partial}{\partial u}\left(\lim_{r\to\infty}\tilde{\gamma}_{ab}\right) = 0,$$

so that we have

$$\lim_{r\to\infty}\tilde{\gamma}_{ab} = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$$

for all $u \in (C_1, C_2)$. As discussed in § 2 we can achieve

$$\lim_{r\to\infty} U^a = 0$$

by appropriately propagating the coordinates x^a away from the surface $u = 0$.

The above construction shows that the Bondi coordinates above are uniquely determined by the choice of a function $\breve{u} = u|_{\mathscr{I}}$. Let us show that the freedom in the choice of $u|_{\mathscr{I}}$ can be considerably reduced if we require

$$\lim_{r\to\infty}\beta = 0. \tag{4.15}$$

To achieve (4.15), let $\bar{u}|_{\mathscr{I}}$ be given by

$$\bar{u}(u, x^a) = \int_0^u \mathrm{e}^{2H(u', x^a)}\,\mathrm{d}u' + \alpha(x^a) \qquad \left(H = \lim_{r\to\infty}\beta\right), \tag{4.16}$$

where $\alpha \in C^\infty(S^2)$ is an arbitrary function. We can now repeat the construction described above of a coordinate system $(\bar{s}, \bar{u}, \bar{x}^a)$ based upon this function $\bar{u}|_{\mathscr{I}}$, obtaining again a metric of the form (4.13) in this coordinate system. Let thus a vacuum metric γ of the form (2.1) be given, and set $\tilde{\gamma}_{\mu\nu} = r^{-2}\gamma_{\mu\nu}$, $x = r^{-1}$, $k^\mu = \tilde{\gamma}^{\mu\nu}\bar{u}_{,\nu}$; from the equality

$$\tilde{\gamma}^{\mu\nu}|_{\mathscr{I}}\partial_\mu\partial_\nu = -2\mathrm{e}^{-2H}\partial_u\partial_x + \breve{h}^{ab}\partial_a\partial_b$$

by construction of k^μ we obtain

$$k^\mu|_{\mathscr{I}}\partial_\mu = -\partial_x - \mathrm{e}^{-2H}\left.\frac{\partial\bar{u}}{\partial x}\right|_{\mathscr{I}}\partial_u + \mathcal{D}^a\bar{u}|_{\mathscr{I}}\partial_a, \tag{4.17}$$

with

$$\left.\frac{\partial\bar{u}}{\partial x}\right|_{\mathscr{I}} = \tfrac{1}{2}|\mathcal{D}\bar{u}|^2_{\breve{h}}|_{\mathscr{I}}, \tag{4.18}$$

where $\mathcal{D}$ denotes the covariant derivative of the metric $\breve{h}$ on S^2. We also have

$$\left.\frac{\partial\bar{x}^a}{\partial u}\right|_{\mathscr{I}} = 0, \quad \left.\frac{\partial\bar{x}^a}{\partial x^b}\right|_{\mathscr{I}} = \delta^a_b, \quad \left.\frac{\partial\bar{x}^a}{\partial x}\right|_{\mathscr{I}} = \mathcal{D}^a\bar{u}|_{\mathscr{I}}. \tag{4.19}$$

From the 'barred' equivalent of (4.14),

$$\bar{r} = \left[\sin^2\theta \det\left(\gamma^{\mu\nu}\frac{\partial\bar{x}^a}{\partial x^\mu}\frac{\partial\bar{x}^b}{\partial x^\nu}\right)\right]^{-1/4},$$

and from what has been said one easily finds

$$\lim_{r\to\infty}\frac{\partial\bar{r}}{\partial r} = 1, \qquad \lim_{r\to\infty}\frac{\partial\bar{r}}{\partial u} = \lim_{r\to\infty}\frac{\partial\bar{r}}{\partial x^a} = 0. \tag{4.20}$$

Equations (4.18)–(4.20) show that in the coordinate system $(\bar{u}, \bar{r}, \bar{x}^a)$ the metric takes the desired form with

$$\lim_{r\to\infty}\beta = 0. \tag{4.21}$$

It is worthwhile mentioning that the only freedom in the choice of coordinates left at this stage is that of the function α in (4.16). Any two coordinate systems (u_1, r_1, x_1^a) and (u_2, r_2, x_2^a) which satisfy our requirements will have the property that

$$\alpha \equiv (u_1 - u_2)|_{\mathscr{I}}$$

is a function of x^a only, and the coordinate system (u_2, r_2, x_2^a) is defined uniquely by α and by (u_1, r_1, x_1^a). It follows that for polyhomogeneous metrics the asymptotic symmetry group is the BMS group, as in the smooth case.

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Appendix A. Conventions, function spaces

We shall assume that all manifolds we discuss (which will have dimension 2, 3 or 4) are paracompact, connected, Hausdorff and smooth. The summation

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convention is used throughout this paper. Space-time is as usual taken to be a Lorentzian four-dimensional manifold with signature $(-,+,+,+)$ and metric connection $\Gamma^{\lambda}{}_{\mu\nu}$ where Greek indices run from 0 to 3; its Riemann curvature tensor is defined by

$$R^{\alpha}{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^{\alpha}{}_{\beta\delta} - \partial_\delta \Gamma^{\alpha}{}_{\beta\gamma} + \Gamma^{\alpha}{}_{\mu\gamma}\Gamma^{\mu}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\mu\delta}\Gamma^{\mu}{}_{\beta\gamma}$$

and the Ricci tensor and scalar by

$$R_{\beta\delta} = R^{\alpha}{}_{\beta\alpha\delta}, \quad R = g^{\mu\nu}R_{\mu\nu}.$$

With these choices the Einstein equations take the form

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu},$$

with a positive constant κ.

In this paper we are considering asymptotic behaviour near null infinity. We will use $\bar{M}$ to denote a manifold with boundary so that $M \equiv \operatorname{int} \bar{M}$ is a manifold of dimension n and $\partial M \equiv \partial \bar{M}$ is a manifold which will be assumed to have a finite number of connected components ∂M_i. By an abuse of terminology M will also be said to be a manifold with boundary. As usual T_pM will denote the tangent space to M at p; for p in ∂M one has the notion of 'half-tangent space at p' which is defined in a natural obvious way, and we shall still write T_pM for this space.

Throughout the paper x will denote a defining function for ∂M, i.e. a function satisfying $x|_{\partial M} = 0$, $x \geqslant 0$, $\mathrm{d}x(p) \neq 0$ for $p \in \partial M$, and the implication $x(p) = 0 \Rightarrow p \in \partial M$ holds.

We can always choose a finite number of coordinate charts $\phi_j : \mathcal{O}_j \to \mathbb{R}^{n,+} \equiv \{y \in \mathbb{R}^n : y^1 \geqslant 0\}$, $j = 1, \dots, J$, covering a neighbourhood of ∂M such that $y^1 = x$. When referring to local coordinates we shall implicitly assume that $y^1 = x$, and we shall use the letter v to denote the coordinates $y^2, \dots, y^n$;

$$v^A = y^A, \quad A = 2, \dots, n.$$

Thus $y = (x, v)$. The standard Schwarz multi-index notation is used throughout; thus if $\alpha = (\alpha_1, \dots \alpha_n)$, then

$$\partial^\alpha = \partial^\alpha_y = \partial^{\alpha_1}_{y^1} \cdots \partial^{\alpha_n}_{y^n} = \partial^{\alpha_1}_x \partial^{\alpha_2}_{y^2} \cdots \partial^{\alpha_n}_{y^n} = \partial^{\alpha_1}_x \partial^\beta_v,$$

where $\beta = (\alpha_2, \dots, \alpha_n)$.

For $k \in \mathbb{N}_0^\infty \equiv \mathbb{N} \cup \{0\} \cup \{\infty\}$ the spaces $C^k_{\mathrm{loc}}(M)$ are the spaces of functions k-times differentiable on M. We have added the subscript 'loc' to emphasize the fact that a function in $C^k_{\mathrm{loc}}(M)$ need *not* extend to the boundary of M (in this respect the subscript 'loc' does not imply the same sense of 'local' as the local coordinates we have just defined); similarly for $k \geqslant 1$ even if the function itself extends by continuity to ∂M then its derivatives do not have to extend, etc. We use the symbol $C^k(\bar{M})$ for the Banach spaces of functions differentiable k-times on M such that f and its derivatives up to order k can be extended to *continuous* functions on $\bar{M}$, and equipped with the supremum norm.

Let f_i be a sequence of functions in $C^\infty_{\mathrm{loc}}(M)$ and let $x_1 > 0$; we suppose that, given $N \in \mathbb{N}$, there is a sequence $s_{i,N} \underset{i\to\infty}{\longrightarrow} \infty$ and some constants $C_{i,N}$ such that for all $|\alpha| \leqslant N$ and for all $0 < x \leqslant x_1$,

$$|\partial^\alpha_y f_i| \leqslant C_{i,N} x^{s_{i,N}},$$

To express the notion of successive approximations good to all powers of x and for all derivatives in a precise sense, we shall write

$$f \sim \sum_{i=0}^{\infty} f_i$$

if for every $n, m \in \mathbb{N}$ there exists $N \in \mathbb{N}$ and a constant $C(n,m)$ such that for all $|\alpha| \leqslant m$ and for $0 < x \leqslant x_1$

$$\left| \partial_y^\alpha \left(f - \sum_{i=0}^{N} f_i \right) \right| \leqslant C(n,m)\, x^n.$$

Consider a sequence $\{N_j\}_{j=0}^{\infty}$, $N_j \in \mathbb{N}_0$. f will be said to be polyhomogeneous if $f \in C^\infty_{\mathrm{loc}}(M)$ and there exists a sequence of functions $f_{jk} \in C^\infty(\bar{M})$ such that

$$f \sim \sum_{j=0}^{\infty} \sum_{k=0}^{N_j} f_{jk}\, x^j \log^k x. \tag{A 1}$$

We write $f \in \mathcal{A}^{\{N_j\}}$, and define $\mathcal{A}_{phg} \equiv \cup_{\{N_j\}} \mathcal{A}^{\{N_j\}}$.

Remark. As formulated here, the f_{ij} may depend upon x. To avoid this would require the introduction of local coordinates near the boundary, a specialization which we do not yet wish to make. However, once we do fix a coordinate system, then we can Taylor expand each f_{ij} with respect to x to any finite order, so each f_{ij} has a polyhomogeneous expansion with no log terms, and obtain an expansion for f with some (other) functions f_{ij} which depend only upon the coordinates v.

A function $f(r,v)$ defined on an open set of the form $\mathcal{O} = \{(r,v) : r \in (r_0,\infty), v \in Q\}$ for some suitable set Q will be said to be in $C^k(\bar{\mathcal{O}})$, $0 \leqslant k \leqslant \infty$, if $f(1/x, v) \in C^k(\bar{\mathcal{U}})$, where $\mathcal{U} = \{(x,v) : x \in (0, 1/r_0), v \in Q\}$. Similarly f will be said to be polyhomogeneous on $\mathcal{O}$ if $f(1/x,v)$ is polyhomogeneous near $x = 0$ on $\mathcal{U}$.

Let F be a function space over M. A tensor field $X = (X^\alpha{}_\beta)$, where α, β are some multi-indices, $|\alpha| = r$, $|\beta| = s$, will be said to belong to F if in local coordinates as described at the beginning of this section the components $X^\alpha{}_\beta$ of X are in F.

Let F be a function space. We shall write that $f \in r^\alpha \log^\beta r F$ if $r^{-\alpha} \log^{-\beta} r f \in F$.

Let F_1, F_2 be function spaces. We shall write that $f \in F_1 + F_2$ if there exist $f_a \in F_a$, $a = 1, 2$, such that $f = f_1 + f_2$.

The following observations are useful when proving Proposition 2.1:

Proposition A.1.

$$f \in r^\alpha \log^\beta r\, \mathcal{A}_{phg}, \quad g \in r^\mu \log^\nu r\, \mathcal{A}_{phg} \Longrightarrow fg \in r^{\alpha+\mu} \log^{\beta+\nu} r\, \mathcal{A}_{phg}; \tag{A 2}$$

$$F \in C^\infty(\mathbb{R}), \quad f \in C^\infty(\bar{\mathcal{N}}) + \mathcal{A}_{phg}/r \Longrightarrow F(f) \in \mathcal{A}_{phg}; \tag{A 3}$$

$$i \in \mathbb{Z}, \quad f \in r^i\, \mathcal{A}_{phg} \Longrightarrow \forall j \in \mathbb{N},\ \partial_r^j f \in r^{i-j}\, \mathcal{A}_{phg}, \tag{A 4}$$

$$\forall\, \alpha, \quad \partial_v^\alpha f \in r^i \mathcal{A}_{phg} \tag{A 5}$$

$$i \in \mathbb{Z}, \quad f \in r^i\, \mathcal{A}_{phg} \Longrightarrow \int_R^r f \in r^{i+1}\, \mathcal{A}_{phg}. \tag{A 6}$$

Proof. Equation (A 2) is easily proved starting with the following elementary observations: $f, g \in \mathcal{A}_{phg} \Longrightarrow f+g \in \mathcal{A}_{phg}$; $f \in r^\alpha \log^\beta r\, C^\infty(\partial M)$ (note the Remark above), $g \in \mathcal{A}_{phg} \Longrightarrow fg \in r^\alpha \log^\beta r \mathcal{A}_{phg}$. Equation (A 3) follows similarly from (A 2) by Taylor expanding F. Equation (A 4) and (A 5) are elementary. Equation (A 6) follows by linearity from

$$\int r^i \log^j r \,\mathrm{d}r = \begin{cases} \displaystyle\sum_{k=0}^{j} C_{ijk} r^{i+1} \log^k r, & i \neq -1, \\ \displaystyle\frac{\log^{j+1} r}{j+1}, & i = -1, \end{cases}$$

for some coefficients C_{ijk}. ∎

Appendix B. Geodesics in polyhomogeneous metrics

Proposition B.1. *Let g be a polyhomogeneous metric (Lorentzian or Riemannian) on a manifold with boundary M, $g \in \mathcal{A}_{phg} \cap C^0(\bar{M})$.*

(i) For any $p \in \partial M$, $k \in T_p\bar{M}$ there exists $\epsilon > 0$ and a unique geodesic $\Gamma_p(s)$, $s \in [0, \epsilon)$, satisfying

$$\Gamma_p(0) = p, \qquad \dot{\Gamma}_p(0) = k. \tag{B 1}$$

If we write $\Gamma_p = \{y^\mu(s)\}$, then $y^\mu(s)$ are polyhomogeneous functions of s.

(ii) If $\partial M \ni p \to k_p$ is a smooth field, $k \in C^\infty(\partial M)$, then the functions $y^\mu(s, v)$ are polyhomogeneous.

Remark. For a polyhomogeneous metric we have $\partial g \sim \log^{N_1} x$ near ∂M, so that standard results about geodesics do not apply.

Proof. Let us define

$$\psi^\mu(s) = y^\mu(s) - sk^\mu - y_0^\mu,$$

we thus have

$$\frac{\mathrm{d}^2\psi^\mu}{\mathrm{d}s^2} = F^\mu(\psi, \dot{\psi}, s), \tag{B 2}$$

with

$$F^\mu(\psi, \chi, s) = \Gamma^\mu_{\alpha\beta}(y_0^\nu + sk^\nu + \psi^\nu)\,\chi^\alpha \chi^\beta. \tag{B 3}$$

Let $\alpha \in (0, \frac{1}{2})$ and let $\epsilon > 0$ be a number to be determined later. Consider the space

$$X_\epsilon = \{\psi^\mu, \chi^\mu \in C([0, \epsilon])\}$$

with the norm

$$\|(\psi, \chi)\|_{X_\epsilon} = \sup_{s\in[0,\epsilon]} |s^{-\alpha}\chi^\mu(s)| + \sup_{s\in[0,\epsilon]} |s^{-\alpha-1}\psi^\mu(s)|.$$

Let $T : X_\epsilon \to X_\epsilon$ be the map

$$X_\epsilon \ni (\psi^\mu, \chi^\mu) \longrightarrow T[\psi, \chi] = \left(\int_0^s \chi^\mu(s)\,\mathrm{d}s, \int_0^s F^\mu(\psi, \chi, s)\,\mathrm{d}s\right), \tag{B 4}$$

F^μ given by (B 3). Clearly a fixed point of T satisfies (B 1)–(B 2). From the estimates

$$|\Gamma| + |\partial_v \Gamma| \leqslant C \log^N x, \tag{B 5}$$

$$|\partial_x \Gamma| \leqslant C x^{-1} \log^N x, \tag{B 6}$$

for some N, it is easily seen that one can choose a constant K and an $\epsilon > 0$ such that T is a contraction mapping from a ball around $(0,0)$ in X_ϵ of radius K into itself, and the result follows by the contraction mapping principle. Once the solution is known to exist, polyhomogeneity immediately follows from the equation

$$(\psi, \chi) = T[\psi, \chi].$$

If $k^\mu(v)$ is a smooth function of $v \in \partial M$, then by considering the equations satisfied by $\partial_v^\alpha y^\mu$ (note that for a polyhomogeneous metric $\partial_v^\alpha \Gamma$ satisfies the same estimates (B 5)–(B 6) as Γ itself) one obtains the result by an argument similar to the one above. ■

Let us finally point out that part (i) of Proposition B.1 is still true under the rather weaker hypotheses

$$|g^{\mu\nu}| + |g_{\mu\nu}| + x^{1-\beta}|\partial_\sigma g_{\mu\nu}| + x^{2-\beta}|\partial_\sigma \partial_\rho g_{\mu\nu}| \leqslant C, \tag{B 7}$$

for some constant C, with any $\beta > 0$. The same proof as above goes through, except that the exponent α in the norm $\|(\psi, \chi)\|_{X_\epsilon}$ has to be chosen to lie in $(0, \beta)$. Gauss coordinates can be constructed for metrics satisfying (B 7) provided that one moreover has

$$x^{1-\beta}|\partial_A \partial_\rho g_{\mu\nu}| + x^{2-\beta}|\partial_A \partial_\rho \partial_\sigma g_{\mu\nu}| \leqslant C, \tag{B 8}$$

where ∂_A are derivatives in directions tangent to ∂M, while ∂_ρ, etc., denotes all partial derivatives.

Appendix C. The Einstein field equations for the metric (2.1)

Using SHEEP we have derived the Einstein equations for a metric of the form (2.1). The metric coefficients have been further parametrized as in Proposition 3.2 with the small changes that, to aid comparison with van der Burg (1966), U^θ is written as U and U^ϕ is written as $W \operatorname{cosec} \theta$. Derivatives with respect to the coordinates, which are numbered by

$$(x_0, x_1, x_2, x_3) = (u, r, \theta, \phi)$$

are indicated by subscripts. The equations one obtains coincide with those of the appendix in van der Burg (1966), except for some misprints listed below. If all terms were moved to the left-hand sides, the left sides would be $rR_{11}/4$ in (C 1), $2r^2 R_{12}$ in (C 2), $2r^2 R_{13} \operatorname{cosec}\theta$ in (C 3), $-\mathrm{e}^{2\beta}(h^{ab}R_{ab})/2$ in (C 4), $\mathrm{e}^{2\beta}(\mathrm{e}^{-2\gamma}R_{22} - \mathrm{e}^{2\gamma}\sin^{-2}\theta R_{33})/4r$ in (C 5) and $\mathrm{e}^{2\beta}(R_{23} - (h^{ab}R_{ab})h_{23}/2)/(2r\cosh 2\delta \sin\theta)$ in (C 6).

The misprints in the corresponding equations of van der Burg (1966) are:

(i) on p. 121, at the end of the second line of the second equation, a right parenthesis is missing;

(ii) on p. 122, in the fourth line of the equation the signs of the terms which contain β_{23} and $\beta_2\beta_3$ should be reversed, and

(iii) in the fifth line of the same equation the factor r should be replaced by r^3.

$$\beta_1 = \tfrac{1}{2} r(\gamma_1^2 \cosh^2 2\delta + \delta_1^2), \tag{C 1}$$

$$\begin{aligned}(r^4 e^{-2\beta}(e^{2\gamma} U_1 \cosh 2\delta + W_1 \sinh 2\delta))_1 \\ = 2r^2(\beta_{12} + 2\delta_1\delta_2 - 2r^{-1}\beta_2 - 4\gamma_1\delta_2 \cosh 2\delta \sinh 2\delta \\ -(\gamma_{12} - 2\gamma_1\gamma_2 + 2\gamma_1 \cot\theta)\cosh^2 2\delta) \\ +2r^2 e^{2\gamma} \operatorname{cosec}\theta(-\delta_{13} - 2\delta_1\gamma_3 + (\gamma_{13} + 2\gamma_1\gamma_3)\cosh 2\delta \sinh 2\delta \\ +2\gamma_1\delta_3(1 + 2\sinh^2 2\delta)),\end{aligned} \tag{C 2}$$

$$\begin{aligned}(r^4 e^{-2\beta}(U_1 \sinh 2\delta + e^{-2\gamma} W_1 \cosh 2\delta))_1 \\ = 2r^2 e^{-2\gamma}(-\delta_{12} + 2\delta_1\gamma_2 - 2\delta_1 \cot\theta \\ -(\gamma_{12} - 2\gamma_1\gamma_2 + 2\gamma_1 \cot\theta)\cosh 2\delta \sinh 2\delta - 2\gamma_1\delta_2(1 + 2\sinh^2 2\delta)) \\ +2r^2 \operatorname{cosec}\theta(\beta_{13} + 2\delta_1\delta_3 - 2r^{-1}\beta_3 + 4\gamma_1\delta_3 \cosh 2\delta \sinh 2\delta \\ +(\gamma_{13} + 2\gamma_1\gamma_3)\cosh^2 2\delta),\end{aligned} \tag{C 3}$$

$$\begin{aligned}V_1 = {} & 2e^{2\beta} \operatorname{cosec}\theta((\beta_{23} + \beta_2\beta_3 + 2\delta_2\delta_3)\sinh 2\delta + (\delta_{23} + \delta_3 \cot\theta + \delta_2\gamma_3 - \gamma_2\delta_3 \\ & +\delta_2\beta_3 + \beta_2\delta_3)\cosh 2\delta) \\ & -e^{2\beta-2\gamma}((\beta_{22} + \beta_2^2 + \beta_2 \cot\theta + 2\gamma_2^2 + 2\delta_2^2 - 1 - \gamma_{22} \\ & -3\gamma_2 \cot\theta - 2\beta_2\gamma_2)\cosh 2\delta + (\delta_{22} + 3\delta_2 \cot\theta + 2\beta_2\delta_2 - 4\gamma_2\delta_2)\sinh 2\delta) \\ & -e^{2\beta+2\gamma} \operatorname{cosec}^2\theta((\beta_{33} + \beta_3^2 + 2\gamma_3^2 + 2\delta_3^2 + \gamma_{33} + 2\beta_3\gamma_3)\cosh 2\delta \\ & +(\delta_{33} + 2\beta_3\delta_3 + 4\gamma_3\delta_3)\sinh 2\delta) \\ & -\tfrac{1}{4} r^4 e^{-2\beta}((e^{2\gamma} U_1^2 + e^{-2\gamma} W_1^2)\cosh 2\delta + 2U_1 W_1 \sinh 2\delta) \\ & +\tfrac{1}{2} r(rU_{12} + rU_1 \cot\theta + 4U_2 + 4U \cot\theta) + \tfrac{1}{2} r^2 \operatorname{cosec}\theta(W_{13} + 4W_3),\end{aligned} \tag{C 4}$$

$$\begin{aligned}(r\gamma)_{01} \cosh 2\delta + 2r(\gamma_0\delta_1 + \delta_0\gamma_1)\sinh 2\delta \\ = \tfrac{1}{2}(\gamma_1 V_1 + \gamma_{11} V + r^{-1}\gamma_1 V)\cosh 2\delta + 2\gamma_1\delta_1 V \sinh 2\delta \\ +\tfrac{1}{8} r^3 e^{-2\beta}(e^{2\gamma} U_1^2 - e^{-2\gamma} W_1^2) + \tfrac{1}{2} r^{-1} e^{2\beta-2\gamma}(\beta_{22} + \beta_2^2 - \beta_2 \cot\theta) \\ -\tfrac{1}{2} r^{-1} e^{2\beta+2\gamma}(\beta_{33} + \beta_3^2)\operatorname{cosec}^2\theta + r^{-1} e^{2\beta}(\beta_2\delta_3 - \beta_3\delta_2)\operatorname{cosec}\theta \\ +\tfrac{1}{4} r e^{2\gamma} \operatorname{cosec}\theta((U_{13} + 2r^{-1}U_3)\sinh 2\delta + 4\delta_1 U_3 \cosh 2\delta) \\ -\tfrac{1}{4} r e^{-2\gamma}((W_{12} - W_1 \cot\theta)\sinh 2\delta + 2r^{-1}(W_2 - W \cot\theta)\sinh 2\delta \\ +4\delta_1(W_2 - W \cot\theta)\cosh 2\delta) \\ -\tfrac{1}{4} r(U_{12} + 2r^{-1}U_2 - U_1 \cot\theta - 2r^{-1}U \cot\theta \\ +4r^{-1}\gamma_2 U + 4\gamma_{12} U + 2\gamma_2 U_1 + 2\gamma_1 U_2 + 2\gamma_1 U \cot\theta)\cosh 2\delta \\ -r(\delta_1 U_2 + 2\gamma_1\delta_2 U + 2\delta_1\gamma_2 U - \delta_1 U \cot\theta)\sinh 2\delta \\ +\tfrac{1}{4} r \operatorname{cosec}\theta(W_{13} + 2r^{-1}W_3 - 4r^{-1}\gamma_3 W - 4\gamma_{13} W - 2\gamma_3 W_1 \\ -2\gamma_1 W_3)\cosh 2\delta \\ +r \operatorname{cosec}\theta(\delta_1 W_3 - 2\delta_1\gamma_3 W - 2\gamma_1\delta_3 W)\sinh 2\delta,\end{aligned} \tag{C 5}$$

$$
\begin{aligned}
(r\delta)_{01} &- 2r\gamma_0\gamma_1 \sinh 2\delta \cosh 2\delta \\
&= \tfrac{1}{2}(\delta_1 V_1 + \delta_{11} V + r^{-1}\delta_1 V - 2\gamma_1^2 V \cosh 2\delta \sinh 2\delta) \\
&\quad -\tfrac{1}{2} r^{-1} e^{2\beta - 2\gamma}(\beta_{22} + \beta_2^2 - \beta_2 \cot\theta)\sinh 2\delta \\
&\quad -\tfrac{1}{2} r^{-1} e^{2\beta + 2\gamma} \operatorname{cosec}^2\theta(\beta_{33} + \beta_3^2)\sinh 2\delta \\
&\quad -r^{-1} e^{2\beta} \operatorname{cosec}\theta(-\beta_{23} - \beta_2\beta_3 + \beta_3\cot\theta + \beta_2\gamma_3 - \gamma_2\beta_3)\cosh 2\delta \\
&\quad +\tfrac{1}{8} r^3 e^{-2\beta}((e^{2\gamma}U_1^2 + e^{-2\gamma}W_1^2)\sinh 2\delta + 2U_1W_1\cosh 2\delta) \\
&\quad -\tfrac{1}{2} r(2\delta_{12}U + 2r^{-1}\delta_2 U + \delta_1 U_2 + \delta_2 U_1 + \delta_1 U \cot\theta \\
&\qquad -2(\gamma_1 U_2 - \gamma_1 U\cot\theta + 4\gamma_1\gamma_2 U)\cosh 2\delta \sinh 2\delta) \\
&\quad -\tfrac{1}{2} r \operatorname{cosec}\theta(2\delta_{13}W + 2r^{-1}\delta_3 W + \delta_1 W_3 + \delta_3 W_1 \\
&\qquad +2(\gamma_1 W_3 - 2\gamma_1\gamma_3 W)\cosh 2\delta \sinh 2\delta) \\
&\quad -\tfrac{1}{4} r e^{-2\gamma}(W_{12} - W_1\cot\theta + 2r^{-1}(W_2 - W\cot\theta) \\
&\qquad -4\gamma_1(W_2 - W\cot\theta)\cosh^2 2\delta) \\
&\quad -\tfrac{1}{4} r e^{2\gamma} \operatorname{cosec}\theta(U_{13} + 2r^{-1}U_3 + 4\gamma_1 U_3 \cosh^2 2\delta). \qquad \text{(C 6)}
\end{aligned}
$$

Appendix D. Expansion in the axisymmetric case

The following formulae give the expansion of the axisymmetric case discussed at the end of § 3. All coefficients, except the last one in each quantity (i.e. γ_4, β_5, U_5 and V_3) are considered to be functions of u and θ only: γ_4, β_5, U_5 and V_3 are written as functions of u, θ and r so that the consistency of the approximation can be checked by looking at the first neglected terms in the equations of Appendix C. The exponentials of β and γ are expanded to order r^{-4}.

As a result of the arguments of § 2, we know we can expand γ as

$$\gamma = c(v)r^{-1} + \gamma_2(v)r^{-2} + \gamma_{3,1}(v)r^{-3}\log r + \gamma_3(v)r^{-3} + \gamma_4(r,v)r^{-4},$$

where we use (v) to denote (u, x^a), and the dependence of γ_4 on r allows for $\log r$ terms there. To avoid confusion of subscripts in what follows we use the form $f_{,x}$ for the partial derivatives, where x is a variable name, not a number.

Substitution in (C 1) yields

$$\beta = -\tfrac{1}{4}c^2r^{-2} - \tfrac{2}{3}c\gamma_2 r^{-3} - \tfrac{3}{4}c\gamma_{3,1}r^{-4}\log r + (\tfrac{1}{16}c\gamma_{3,1} - \tfrac{3}{4}c\gamma_3 - \tfrac{1}{2}\gamma_2^2)r^{-4} + \beta_5 r^{-5}.$$

Putting this into (C 2) gives

$$
\begin{aligned}
U = &-(2c\cot\theta + c_{,\theta})r^{-2} - (\tfrac{8}{3}\gamma_2\cot\theta + \tfrac{4}{3}\gamma_{2,\theta})r^{-3}\log r + r^{-3}U_3 \\
&+(2c\gamma_{2,\theta} + 4c\gamma_2\cot\theta + \tfrac{3}{2}\gamma_{3,1,\theta} + 3\gamma_{3,1}\cot\theta)r^{-4}\log r \\
&+(\tfrac{5}{2}c^3\cot\theta + \tfrac{5}{4}c^2c_{,\theta} + \tfrac{5}{3}c\gamma_2\cot\theta - \tfrac{3}{2}cU_3 + \tfrac{1}{2}c\gamma_{2,\theta} + \tfrac{2}{3}\gamma_2 c_{,\theta} \\
&+\tfrac{11}{4}\gamma_{3,1}\cot\theta + 3\gamma_3\cot\theta + \tfrac{11}{8}\gamma_{3,1,\theta} + \tfrac{3}{2}\gamma_{3,\theta})r^{-4} + r^{-5}U_5,
\end{aligned}
$$

where U_3 is as yet arbitrary. Its u-derivative is given by the R_{02} Einstein equation, and on substituting van der Burg's form for the coefficient

$$U_3 = 4c^2\cot\theta + 3cc_{,\theta} + 2N$$

we find agreement with his equation for $N_{,u}$.

Next we use (C 4) to obtain

$$\begin{aligned}V = {} & r - 2M + r^{-1}\log r(2\cot\theta\gamma_{2,\theta} - \tfrac{4}{3}\gamma_2 + \tfrac{2}{3}\gamma_{2,\theta\theta})\\ & + r^{-1}(\gamma_{2,\theta}\cot\theta + \tfrac{5}{2}(c_{,\theta})^2 + \tfrac{1}{3}\gamma_{2,\theta\theta} - \tfrac{1}{2}U_{3,\theta} + 4c^2\cot^2\theta - \tfrac{3}{2}c^2 - \tfrac{2}{3}\gamma_2\\ & \quad + \tfrac{3}{2}cc_{,\theta\theta} + \tfrac{19}{2}cc_{,\theta}\cot\theta - \tfrac{1}{2}U_3\cot\theta)\\ & + r^{-2}\log r(8c\gamma_2\cot^2\theta + 4c\gamma_{2,\theta}\cot\theta + 4\gamma_2 c_{,\theta}\cot\theta + 2c_{,\theta}\gamma_{2,\theta}\\ & \quad - \tfrac{1}{2}\gamma_{3,1,\theta\theta} - \tfrac{3}{2}\gamma_{3,1,\theta}\cot\theta + \gamma_{3,1})\\ & + r^{-2}(4c^3\cot^2\theta + \tfrac{1}{6}c^3 + \tfrac{9}{4}c^2c_{,\theta}\cot\theta - \tfrac{1}{4}c^2c_{,\theta\theta} + \tfrac{11}{6}c\gamma_{2,\theta}\cot\theta + \tfrac{1}{6}c\gamma_{2,\theta\theta}\\ & \quad + \tfrac{7}{3}\gamma_2 c_{,\theta}\cot\theta + \tfrac{2}{3}\gamma_2 c_{,\theta\theta} + \tfrac{7}{6}c_{,\theta}\gamma_{2,\theta} + \tfrac{4}{3}c\gamma_2\cot^2\theta - \tfrac{1}{3}c\gamma_2 - 3cU_3\cot\theta\\ & \quad - \tfrac{3}{2}U_3c_{,\theta} - \tfrac{5}{8}\gamma_{3,1,\theta\theta} - \tfrac{15}{8}\gamma_{3,1,\theta}\cot\theta + \tfrac{5}{4}\gamma_{3,1} - \tfrac{1}{2}\gamma_{3,\theta\theta} - \tfrac{3}{2}\gamma_{3,\theta}\cot\theta + \gamma_3)\\ & + r^{-3}V_3,\end{aligned}$$

where M is as yet undetermined, but will have a u-derivative given by the R_{00} equation as

$$M_{,u} = \tfrac{3}{2}\cot\theta c_{,u\theta} - c_{,u}{}^2 - c_{,u} + \tfrac{1}{2}c_{,u\theta\theta}.$$

On substituting all these expansions into the equation (C 5) we find that $c_{,u}$ is undetermined, and $\gamma_{2,u} = 0$ as expected. $\gamma_{3,1,u}$ is then u-independent so we can integrate with respect to u and get

$$\gamma_{3,1} = \gamma^0_{3,1} + (\tfrac{1}{6}\gamma_{2,\theta\theta} + \tfrac{1}{6}\gamma_{2,\theta}\cot\theta - \tfrac{2}{3}\gamma_2\cot^2\theta - \tfrac{1}{3}\gamma_2)u,$$

where $\gamma^0_{3,1}$ is a freely specifiable function of θ.

The next term in (C 5) gives the u-derivative of γ_3 as

$$\begin{aligned}\gamma_{3,u} = {} & \tfrac{3}{8}cc_{,\theta\theta} + \tfrac{5}{8}cc_{,\theta}\cot\theta + \tfrac{3}{8}(c_{,\theta})^2 - c^2\cot^2\theta - \tfrac{1}{2}c^2 + \tfrac{1}{2}cM\\ & - \tfrac{1}{12}\gamma_{2,\theta\theta} - \tfrac{1}{12}\gamma_{2,\theta}\cot\theta + \tfrac{1}{3}\gamma_2\cot^2\theta - \tfrac{1}{3}\gamma_2 - \tfrac{1}{8}U_{3,\theta} + \tfrac{1}{8}U_3\cot\theta\end{aligned}$$

and we cannot integrate this explicitly since we do not know the u dependence of c and hence of M and U_3. We can check, using

$$\gamma_3 = C - \tfrac{1}{6}c^3$$

to get van der Burg's form of the coefficient, that our result agrees with his when $\gamma_2 = 0$.

Finally we can solve the next order in (C 5) for $\gamma_{4,u}$ and find that the result contains $\log r$ terms but no higher powers of $\log r$, and that

$$\gamma_{4,u}\sin^3\theta = G + F_{,\theta},$$

where G is as in (3.12) and

$$\begin{aligned}F = {} & \log r((-\tfrac{1}{24}\gamma_{2,\theta\theta\theta}\sin^3\theta + \tfrac{1}{24}\gamma_{2,\theta\theta}\cos\theta\sin^2\theta + \tfrac{7}{24}\gamma_{2,\theta}\cos^2\theta\sin\theta\\ & \quad + \tfrac{1}{8}\gamma_{2,\theta}\sin^3\theta - \tfrac{2}{3}\gamma_2\cos^3\theta - \tfrac{1}{2}\gamma_2\cos\theta\sin^2\theta)u\\ & \quad - \tfrac{2}{3}c\gamma_2\cos\theta\sin^2\theta - \tfrac{1}{3}c\gamma_{2,\theta}\sin^3\theta - \tfrac{1}{4}(\gamma^0_{3,1})_{,\theta}\sin^3\theta + \tfrac{1}{2}\gamma^0_{3,1}\cos\theta\sin^2\theta)\\ & + (-\tfrac{1}{32}\gamma_{2,\theta\theta\theta}\sin^3\theta + \tfrac{1}{32}\gamma_{2,\theta\theta}\cos\theta\sin^2\theta + \tfrac{7}{32}\gamma_{2,\theta}\cos^2\theta\sin\theta\\ & \quad + \tfrac{29}{96}\gamma_{2,\theta}\sin^3\theta - \tfrac{1}{2}\gamma_2\cos^3\theta - \tfrac{19}{24}\gamma_2\cos\theta\sin^2\theta)u\\ & - \tfrac{1}{4}\gamma_{3,\theta}\sin^3\theta + \tfrac{1}{2}\gamma_3\cos\theta\sin^2\theta + \tfrac{2}{3}\gamma_2 c_{,\theta}\sin^3\theta + \tfrac{1}{12}c\gamma_{2,\theta}\sin^3\theta\\ & + \tfrac{5}{6}c\gamma_2\cos\theta\sin^2\theta - \tfrac{11}{12}c^3\cos\theta\sin^2\theta - \tfrac{7}{8}c^2c_{,\theta}\sin^3\theta + \tfrac{1}{4}cU_3\sin^3\theta\\ & - \tfrac{3}{16}(\gamma^0_{3,1})_{,\theta}\sin^3\theta + \tfrac{3}{8}\gamma^0_{3,1}\cos\theta\sin^2\theta.\end{aligned}$$

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Since G contains no $\log r$ terms, it follows that $\mathcal{Q}$ as defined in (3.13) is conserved. (We may again note that if $\gamma_2 \equiv 0 \equiv \gamma_{3,1}$ this agrees with van der Burg (1966).)

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